Construction of Orthogonal Wavelets Using

Fractal Interpolation Functions

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Abstract

Fractal interpolation functions are used to construct a compactly supported continuous, orthogonal wavelet basis spanning $L^2(\mathbb{R})$. The wavelets share many of the properties normally associated with spline wavelets in particular linear phase.

Key Words: Wavelets, fractal interpolation functions, linear phase

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I. Introduction

A wavelet basis of some function space, for example $L^2(\mathbb{R})$, is obtained by considering translates and dilates of one or several suitable functions [1, 5, 16, 17, 19]. Much of the recent interest in these bases has stemmed from the fact that they can be built having various useful properties such as continuity, orthogonality, compact support, vanishing moments, etc. Most of the wavelets that have been investigated to date can be constructed using the notion of multiresolution analysis [16, 17]. Let ϕ be a function in $L^2(\mathbb{R})$ and set $\phi_{k,j}(x) = \phi(2^k x - j)$. For each $k \in \mathbb{Z}$, denote by V_k the L^2–closure of the algebraic span of $\{\phi_{k,j} : j \in \mathbb{Z}\}$. The function ϕ is said to generate a multiresolution analysis if the following conditions are satisfied

(i) $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$;

(ii) $\mathrm{clos}_{L^2}\left(\bigcup_{k\in\mathbb{Z}} V_k\right) = L^2$;

(iii) $\bigcap_{k\in\mathbb{Z}} V_k = \{0\}$; and

(iv) $\{\phi_{0,j} : j \in \mathbb{Z}\}$ is an Riesz basis for V_0.

Suppose ϕ generates a multiresolution analysis then ϕ is called a scaling function and it will satisfy the two scale dilation equation

$$\phi(x) = \sum c_j \phi(2x - j). \tag{1.1}$$

If W_0 is the orthogonal complement of V_0 in V_1 there exists a $\psi \in L^2(\mathbb{R})$ such that $\overline{\mathrm{span}\{\psi(\cdot - i), i \in \mathbb{Z}\}} = W_0$, [16, 17]. Furthermore the function ψ satisfies the equation

$$\psi(x) = \sum d_j \phi(2x - j). \tag{1.2}$$

If $\{\phi_{0,k}\}$ is an orthonormal basis for V_0, the formula

$$d_j = (-1)^j c_{1-j}, \tag{1.3}$$

gives a simple way of computing the coefficients of equation (1.2). The celebrated work of Daubechies [5] gives explicit construction of finite sequences of coefficients $\{c_j\}$ which give solutions of equation (1.1) that are orthogonal, compactly supported and have varying degrees of smoothness.

Recently Hardin, Kessler and Massopust [13] showed that certain classes of fractal interpolation functions (FIF) also generate a multiresolution analysis of $L^2(\mathbb{R})$. This multiresolution analysis has certain geometric features that are similar to the multiresolution analysis generated by splines [4]. These results have been generalized to several dimensions in [8] and [9]. In [10] scaling functions were exhibited that form an orthonormal basis for the V_0 given in [10]. Here we continue the investigation of the multiresolution analysis arising from FIF and construct orthogonal, compactly supported continuous wavelets. Wavelets with varying orders of differentiability will be considered in a later paper [7]. These wavelets fall outside the class constructed in [5] and the multiresolution analysis from which they arise yields several scaling functions instead of just one. In this case (1.1) takes the form

$$\Phi(x) = \sum C_j \Phi(2x - j), \tag{1.4}$$

where each C_j is a square matrix the size of which is determined by the number of scaling functions. Multiresolution analyses based upon several scaling functions have also appeared in the work of Micchelli [18], Goodman, Lee, and Wang [11], Goodman and Lee [12], Jia and Shen [14],and Loïc [15].

We proceed as follows: in Section 2 we review the relevant facts on FIF and the multiresolution analysis arising from these function spaces. Then in Section 3 we exhibit and solve the equations that give scaling functions whose dilates form an orthonormal basis for a certain V_0. We also examine the smoothness of these scaling functions and exhibit their Fourier transforms. In Section 4, we use the scaling functions constructed above to find compactly supported, continuous, orthogonal wavelets. We investigate the support properties of these wavelets and discuss how to convert them into a wavelet basis for $L^2[0,1]$. Finally in Section 5 we extend the methods developed in Sections 3 and 4 to integer scalings other than two. For these cases there are a number of parameters that are free to be specified. We examine this in the case of scaling by three and exhibit a one parameter family of symmetric scaling functions.

II. Fractal Interpolation Functions

Let $I = [0,1]$, and $B(I)$ denote the Banach space of bounded real–valued functions on I with the $\|\cdot\|_\infty$ and $C(I) \subset B(I)$ the space of real-valued functions continuous on I. Let $u_i : [0,1) \to [0,1)$ and $v_i : [0,1) \times \mathbb{R} \to \mathbb{R}$, $i = 0, 1, \dots N-1$ be as follows

$$u_i(x) = \frac{1}{N}(x+i), \tag{2.1}$$

$$v_i(x,y) = \lambda_i(x) + s_i y, \tag{2.2}$$

where $\lambda_i(x) \in \Pi_m$, the set of polynomials with degree at most m. It will always be assumed that $s = \max|s_i| < 1$. Let $I_i = u_i([0,1)) = \left[\frac{i}{N}, \frac{i+1}{N}\right)$ for $i = 0, 1, 2, \dots, N-1$, $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_{N-1})$, and define $\Phi_\lambda : B(I) \to B(I)$ by

$$(\Phi_\lambda f)(x) = v_i(u_i^{-1}(x), f(u_i^{-1}(x))), \tag{2.3}$$

for $x \in I_i$, $i = 0, 1, \dots, N-1$. Note that $f_\lambda(0) = \lambda_0(0)/(1-s_0)$, and $f_\lambda(1^-) = \lambda_{N-1}(1^-)/(1-s_{N-1})$.

Equations (2.2) and (2.3) imply that Φ_λ is a contraction on $B(I)$ with contractivity s thus,

$$\|\Phi_\lambda f - \Phi_\lambda g\|_\infty \le s\|f-g\|_\infty, \tag{2.4}$$

and so Φ_λ has a unique attractive fixed point $f_\lambda \in B(I)$. In the event that Φ_λ satisfies the join-up conditions

$$v_{i+1}(0, f_\lambda(0)) = v_i(1^-, f_\lambda(1^-)), \qquad i = 0, 1, \dots, N-2, \tag{2.5}$$

then f_λ is continuous and is called a fractal interpolation function (Barnsley [2]). In general, $G = \text{graph } f_\lambda$ is typically a fractal set in $\mathbb{R}^2$ made up of images of itself. To see this let $w_i : [0,1) \times R \to [0,1) \times R$ be given by

$$w_i(x,y) = (u_i(x), v_i(x,y)),$$

for $i = 0, 1, \dots, N-1$. Then (2.3) implies that

$$G = \bigcup_{j=0}^{N-1} w_j(G).$$

Let $\beta = \bigotimes_{j=0}^{N-1} \Pi_m$ and $\lambda \in \beta$ then the following theorem gives the basic correspondence between elements in β and functions in $B(I)$.

Theorem 2.1. [10, 12] *The mapping* $\lambda \overset{\theta}{\mapsto} f_\lambda$ *is a linear isomorphism from* β *to* $\theta(\beta)$.

We will be interested in the case when $m = 1$. If f_λ satisfies equation (2.5) then $f_\lambda \in C(I)$ and the space $\theta\left(\bigotimes_{j=0}^{N-1} \Pi_1\right) \cap C(I) = S_0$ is seen to be $N+1$ dimensional. Thus each element $g \in S_0$ is completely determined by $g(i/N)$, $i = 0, 1, \ldots, N$. This allows us to view θ in a slightly different manner. Given $\bar{y} \in R^{N+1}$ let $f_{\bar{y}}$ be the unique element of S_0 passing through the points $\left(\frac{i}{N}, y_i\right)$, $i = 0, 1, \ldots, N$. We shall call functions $f_{\bar{y}} \in S_0$ affine fractal interpolation functions (AFIF).

Corollary 2.2. *The map* $\theta : R^{N+1} \to S_0$ *is a linear isomorphism.*

The fact that S_0 is isomorphic to R^{N+1} adds a geometric component to the multiresolution analysis associated with FIF and will play an important role in our construction of wavelets. Let $C_b(\mathbb{R})$ be the space of bounded continuous functions on $\mathbb{R}$ Fix $s_0, s_1, \ldots, s_{N-1}$, let $\tilde{V}_0 = \{f : f|_{[i,i+1)}$ is an AFIF$\} \cap C_b(\mathbb{R}) \cap L^2(\mathbb{R})$ and define $f \in \tilde{V}_k \Leftrightarrow f(N^{-k}\cdot) \in \tilde{V}_0$. Then it was shown in [10] and [13] that the sequence $\{\tilde{V}_i\}$ has the following properties,

(a) $\ldots \tilde{V}_{-1} \subset \tilde{V}_0 \subset \tilde{V}_1 \ldots$,

(b) $\bigcap_{k \in R} \tilde{V}_k = \{0\}$.

Note that in contrast with [16] and [17], the spaces $\{\tilde{V}_i\}$ given above are defined independently of any particular scaling functions which is one of the several properties these spaces share with the spline spaces S_d^r with integer knots (see [10],[11]). In fact $\tilde{V}_0$ is spanned by sets formed from the integer translates of several scaling functions.

We say a multiresolution analysis is *continuous* and/or *compactly supported* if it is generated by a finite set of scaling functions $\{\phi^i(x)\}_{i=1}^N$, $\phi^i(x) \in L^2(\mathbb{R})$, $i = 1 \ldots N$ such that each ϕ^i is continuous and/or compactly supported on $\mathbb{R}$. If $\langle \phi^i(\cdot), \phi^j(\cdot - k)\rangle = \delta_{i,j}\delta_{k,0}$, $i = 1 \ldots N$ then the $\{\phi^i\}$ generate an *orthogonal* multiresolution analysis.

In order to find *orthogonal scaling functions* $\{\phi^i\}_{i=1}^N$ that generate a continuous, compactly supported, orthogonal multiresolution analysis we develop some quadrature formulas for AFIF. Set $I^* = \int_0^1 f_{\bar{y}} g\, dx$ where $f_{\bar{y}} \in S_0$ and $g \in L^2(I)$. Then from (2.3) we find

$$I^* = \sum_{i=0}^{N-1} \int_{[i/N,(i+1)/N]} f_{\bar{y}} g\,dx = \sum_{i=0}^{N-1} \int_{[i/N,(i+1)/N]} v_i(u_i^{-1}(x), f_{\bar{y}}(u_i^{-1}(x))g(x)dx$$
$$= \frac{1}{N} \sum_{i=0}^{N-1} \int_0^1 v_i(x, f_{\bar{y}}(x))g(u_i(x))dx.$$

If we use equation (2.2) along with the assumption that $\lambda_i(x) = a_i x + b_i$, $i = 0, 1, \ldots N-1$, in the above equation we find that

$$\begin{aligned} I^* &= \frac{1}{N} \sum_{i=0}^{N-1} \int_0^1 (a_i x + b_i) g(u_i(x)) dx \\ &\quad + \frac{1}{N} \sum_{i=0}^{N-1} s_i \int_0^1 g(u_i(x)) f_{\bar{y}}(x) dx. \end{aligned} \tag{2.6}$$

If $g = 1$ then (2.6) yields [13]

$$I^* = m_0 = \int_0^1 f_{\bar{y}}(x)dx = \frac{\frac{1}{N}\sum_{i=0}^{N-1}\left(\frac{a_i}{2} + b_i\right)}{1 - \frac{1}{N}\sum_{i=0}^{N-1} s_i}, \tag{2.7}$$

while for $g(x) = x$ we find [13]

$$I^* = m_1 = \int_0^1 x f_{\bar{y}}(x)dx = \frac{\frac{1}{N^2}\sum_{i=0}^{N-1}\left[a_i(i/2 + 1/3) + b_i(i + 1/2) + i s_i m_0\right]}{1 - \frac{1}{N^2}\sum_{i=0}^{N-1} s_i} \tag{2.8}$$

With (2.7) and (2.8) integrals of two fractal functions may be computed. To this end let $\hat{v}_i(x,y) = \hat{\lambda}_i(x) + \hat{s}_i y$ with $\hat{\lambda}_i = \hat{a}_i x + \hat{b}_i$, $\hat{u}_i = u_i$, $i = 0, 1, \ldots, N-1$ and set $g = \hat{f}_{\hat{y}}$. Then [13]

$$\begin{aligned} I^* &= \int_0^1 \hat{f}_{\hat{y}}(x) f_{\bar{y}}(x)dx \\ &= \frac{\frac{1}{N}\sum_{i=0}^{N-1}\left(s_i\hat{a}_i m_1 + \hat{s}_i a_i \hat{m}_1 + s_i\hat{b}_i m_0 + \hat{s}_i b_i \hat{m}_0 + \frac{a_i\hat{a}_i}{3} + \frac{(a_i\hat{b}_i + \hat{a}_i b_i)}{2} + b_i\hat{b}_i\right)}{1 - \frac{1}{N}\sum_{i=0}^{N-1} s_i\hat{s}_i}, \end{aligned} \tag{2.9}$$

where $\hat{m}_0$ and $\hat{m}_1$ are the zeroth and first moments respectively of $\hat{f}_{\hat{y}}$.

Consider the $N+1$ dimensional basis $\{f_{\bar{y}_i}\}_{i=0}^N$ spanning S_0 where $\bar{y}_i = e_i$, $0 < i < N$, $\{e_i\}_{i=0}^N$ being the standard basis in R^{N+1}, $\bar{y}_0 = (1, q_1, \ldots, q_{N-1}, 0)$, and $\bar{y}_N = (0, p_1, \ldots, p_{N-1}, 1)$. The sequences $\{p_i\}$ and $\{q_i\}$ are chosen so that $\langle f_{\bar{y}_0}, f_{\bar{y}_i}\rangle = 0 =$

$\langle f_{\bar{y}_N}, f_{\bar{y}_i}\rangle$ for $i = 1, \ldots, N-1$. Extend $f_{\bar{y}_i}, i = 1, \ldots, N-1$ to be functions in $\tilde{V}_0$ by defining each of them to be equal to zero for $x \notin I$. Let $\{\phi^i\}_{i=0}^{N-2}$ be a sequence of orthogonal functions in $\tilde{V}_0$ obtained from $\{f_{\bar{y}_i}\}_{i=1}^{N-1}$ by the Gram-Schmidt procedure. That these functions are non-zero follows from Corollary 2.2. Set

$$\phi^{N-1}(x) = \begin{cases} f_{\bar{y}_N}(x), & x \in [0,1) \\ f_{\bar{y}_0}(x-1), & x \in [1,2) \\ 0 & \text{elsewhere}, \end{cases} \tag{2.10}$$

and $\hat{\phi}^i(x) = \frac{\phi^i(x)}{\|\phi^i(x)\|_{L^2}}$, then we find the following

Theorem 2.3. *Let $\tilde{V}_0$ and $\phi^i, i = 0, \ldots, N-1$ be as above. Then $\tilde{V}_0 = \text{clos}_{L^2}\text{span}\{\phi^i(\cdot - l) : i = 0, \ldots, N-1, l \in \mathbb{Z}\}$. Furthermore the set $\{\hat{\phi}^i\}_{i=0}^{N-1}$ generates a continuous, compactly supported multiresolution analysis.*

Proof. From Corollary 2.2 and (2.10) we see that each $\phi^i, i = 0, \ldots, N-1$ is compactly supported and is an element of $\tilde{V}_0$. Since every $f \in \tilde{V}_0$ is determined by its values at $\frac{i}{N}, i \in \mathbb{Z}, f$ has a unique expansion in terms of $f_{\bar{y}_i}, i = 1, \ldots, N-1$, and ϕ^{N-1} and their integer translates. Thus every $f \in \tilde{V}_0$ has a unique expansion in terms of $\{\hat{\phi}^i\}_{i=0}^{N-1}$ and their integer translates. In order to show that $\{\hat{\phi}^i\}_{i=0}^{N-1}$ generates a multiresolution analysis we must show that, a) $\{\hat{\phi}^i\}_{i=0}^{N-1}$ and its integer translates form a Riesz basis for $\tilde{V}_0$ and that b) $\text{clos}_{L^2} \bigcup_{k\in Z} \tilde{V}_k = L^2$. Since the set $\{\hat{\phi}^i\}_{i=0}^{N-2}$ and its integer translates is an orthogonal set it follows that we need only show that there exits constants $A, B, 0 < A \leq B < \infty$ such that $\forall c = \{c_i\} \in l^2, A\|c\|_{l^2} \leq \|\sum c_i\hat{\phi}^{N-1}(\cdot - i)\|_{L^2} \leq B\|c\|_{l^2}$. It is easy to show that $B = \sqrt{3}$ provides an upper bound while the lower is obtained by observing that $\|\sum c_i\hat{\phi}^{N-1}(\cdot - i)\|_{L^2}^2 = \sum_i \int_i^{i+1} (c_i\hat{\phi}^{N-1}(x-i) + c_{i-1}\hat{\phi}^{N-1}(x-(i-1)))^2\,dx$. Since $f_{\bar{y}_0}$ and $f_{\bar{y}_N}$ are linearly independent the matrix,

$$K = \begin{pmatrix} \langle f_{\bar{y}_0}, f_{\bar{y}_0}\rangle & \langle f_{\bar{y}_0}, f_{\bar{y}_N}\rangle \\ \langle f_{\bar{y}_0}, f_{\bar{y}_N}\rangle & \langle f_{\bar{y}_N}, f_{\bar{y}_N}\rangle \end{pmatrix},$$

is positive definite. Let λ be the smallest eigenvalue of $\frac{K}{\|\phi^{N-1}\|^2}$ then A can be taken to be equal to $\sqrt{\lambda}$.

To show that $\bigcup_{k\in Z} \tilde{V}_k$ is dense in L^2 we note that for all $x \in \mathbb{R}, 1 = \sum_i(\sum_{j=1}^{N-1} c_i^j f_{\bar{y}_j}(x-i)) + \phi^{N-1}(x-i)$ where $c_i^j = c^j = 1 - p_j - q_j$. Now b) follows from Proposition 3.1 in [10]. $\square$

III. Orthogonal Scaling Functions

We begin by considering AFIF with scaling $N = 2$. By Theorem 2.3 $\{\phi^i\}_{i=0}^{1}$ (note that in this case $\phi^0 = f_{\bar{y}_1}$) and its integer translates span $\tilde{V}_0$ and we look for three mutually orthogonal functions $f_{\bar{y}_0}, f_{\bar{y}_1}, f_{\bar{y}_2}$. From (2.1), (2.2), (2.3), and (2.5) we find that for $f_{\bar{y}_1}$, $\lambda_0 = x$, $\lambda_1 = 1 - x$ for $f_{\bar{y}_2}$, $\lambda_0 = (p_1 - s_0)x$, $\lambda_1 = (1 - s_1 - p_1)x + p_1$ while for $f_{\bar{y}_0}$, $\lambda_0 = (q_1 + s_0 - 1)x + 1 - s_0$, $\lambda_1 = (s_1 - q_1)x + q_1 - s_1$. Substituting these values into (2.9) gives

$$\int_0^1 f_{\bar{y}_1} f_{\bar{y}_2}\, dx = \frac{(4 - 6s_0 + 16p_1 - 2s_1s_0 - 4s_0^2 - 4s_1^2 + 4p_1s_0s_1 + 3s_0^3 + 3s_0s_1^2 - 4p_1s_0^2 - 4p_1s_1^2)}{3(2 - s_0 - s_1)(4 - s_0 - s_1)(2 - s_0^2 - s_1^2)}, \tag{3.1}$$

$$\int_0^1 f_{\bar{y}_1} f_{\bar{y}_0}\, dx = \frac{(4 - 6s_1 + 16q_1 - 2s_1s_0 - 4s_0^2 - 4s_1^2 + 4q_1s_0s_1 + 3s_1^3 + 3s_1s_0^2 - 4q_1s_0^2 - 4q_1s_1^2)}{3(2 - s_0 - s_1)(4 - s_0 - s_1)(2 - s_0^2 - s_1^2)}, \tag{3.2}$$

and

$$\begin{aligned}\int_0^1 f_{\bar{y}_0} f_{\bar{y}_2}\, dx = {} & [4(p_1 + q_1)(2s_0^2 + 2s_1^2 + s_0s_1 - 2) + 8p_1q_1(s_0^2 + s_1^2 - s_0s_1 - 4) + (s_0^2 + s_1^2)^2 - (s_0^2 + s_1^2 + 1)^3 \\ & - 4(s_0 + s_1)^2 + s_0^3(-2 + 2s_1 - 6q_1) + s_1^3(-2 - 6p_1 + 2s_0) + 6q_1s_0(2 - s_1^2) + 6p_1s_1(2 - s_0^2) \\ & + 6s_0 + 6s_1 + 2s_1s_0]/6(-2 + s_0 + s_1)(-4 + s_0 + s_1)(-2 + s_0^2 + s_1^2).\end{aligned} \tag{3.3}$$

Solving (3.1) for p_1 and (3.2) for q_1 yields

$$p_1 = \frac{-(4 - 6s_0 - 2s_1s_0 - 4s_0^2 - 4s_1^2 + 3s_0^3 + 3s_0^2s_1)}{16 + 4s_0s_1 - 4s_0^2 - 4s_1^2}, \tag{3.4}$$

and

$$q_1 = \frac{-(4 - 6s_1 - 2s_1s_0 - 4s_0^2 - 4s_1^2 + 3s_1^3 + 3s_0s_1^2)}{16 + 4s_0s_1 - 4s_0^2 - 4s_1^2}. \tag{3.5}$$

If we substitute (3.4) and (3.5) into (3.3) we find that

$$\int f_{\bar{y}_0} f_{\bar{y}_2}\, dx = \frac{p(s_0, s_1)}{6(16 + 4s_0s_1 - 4s_0^2 - 4s_1^2)^2(-2 + s_0 + s_1)(-4 + s_0 + s_1)(-2 + s_0^2 + s_1^2)},$$

where

$$
\begin{aligned}
p(s_0, s_1) = 2s_1^4 + 6s_1^3 - 7s_1^3 s_0 + 18s_1^2 s_0 - 28s_1^2 - 7s_1 s_0^3 + 18s_1 s_0^2 \\
- 14s_1 s_0 + 12s_1 + 2s_0^4 + 6s_0^3 - 28s_0^2 + 12s_0 + 8.
\end{aligned}
\tag{3.6}
$$

Consequently, we have

Lemma 3.1. *The AFIF $f_{\bar{y}_0}$, $f_{\bar{y}_1}$ and $f_{\bar{y}_2}$ with $\bar{y}_0 = (1, q_1, 0)$, $\bar{y}_1 = (0, 1, 0)$, and $\bar{y}_2 = (0, p_1, 1)$ constitute an orthogonal basis for S_0 only for pairs (s_0, s_1) such that $|s_0| < 1$, $|s_1| < 1$ and $p(s_0, s_1) = 0$.*

Corollary 3.2. *The only pairs (s_0, s_1) such that the basis $\{f_{\bar{x}_1}, f_{\bar{x}_2}, f_{\bar{x}_3}\}$ with $\bar{x}_1 = (0, 1, a)$, $\bar{x}_2 = (0, b, 1)$ and $\bar{x}_3 = (1, c, 0)$ can be made an orthogonal basis for S_0 are those pairs for which $p(s_0, s_1) = 0$. The same is true for bases of the form $\{f_{\bar{z}_1}, f_{\bar{z}_2}, f_{\bar{z}_3}\}$ with $\bar{z}_1 = (a, 1, 0)$, $\bar{z}_2 = (0, b, 1)$ and $\bar{z}_3 = (1, c, 0)$.*

Proof. Let (s_0, s_1) be such that $\{f_{\bar{x}_1}, f_{\bar{x}_2}, f_{\bar{x}_3}\}$ is an orthogonal basis. Suppose $a \neq 0$ otherwise the result follows from Lemma 3.1 and let $\{\hat{f}_{\bar{x}_1}, \hat{f}_{\bar{x}_2}, \hat{f}_{\bar{x}_3}\}$ be the corresponding orthonormal basis. If $\hat{f}_{\bar{x}_1}(1) = a'$ and $\hat{f}_{\bar{x}_2}(1) = b'$ set $w_0 = b'\hat{f}_{\bar{x}_1} - a'\hat{f}_{\bar{x}_2}$ and $w = a'\hat{f}_{\bar{x}_1} + b'\hat{f}_{\bar{x}_2}$ where $a'^2 + b'^2 = 1$. Then $\{w_0, w_1, \hat{f}_{\bar{x}_3}\}$ is an orthonormal basis of S_0 with $w_0(0) = w_0(1) = 0$ and $w_1(0) = 0$. But by Lemma 3.1 this can only happen for values (s_0, s_1) such that $p(s_0, s_1) = 0$. An analogous argument can be applied to the basis $\{f_{\bar{z}_1}, f_{\bar{z}_2}, f_{\bar{z}_3}\}$.

□

From Theorem 2.3 and Lemma 3.1 we have the following

Theorem 3.3. *Suppose that the pair (s_0, s_1) is such that $p(s_0, s_1) = 0$ with $|s_0| < 1$ and $|s_1| < 1$. Then $\hat{\phi}^i$, $i = 0, 1$, generate a continuous, compactly supported, orthogonal multiresolution analysis of $L^2(\mathbb{R})$.*

It follows from Theorem 2.3 that for general pairs (s_0, s_1) with $|s_0| < 1$ and $|s_1| < 1$,

$$
\Phi(x) = \begin{pmatrix} \phi^0 \\ \phi^1 \end{pmatrix} = \sum_{i=0}^{3} C_i \Phi(2x - i). \tag{3.7}
$$

The 2×2 matrices C_i, $i=0,1,2,3$ may be computed by evaluating (3.7) at $x=i/4$, $i=1,2,\ldots,8$. From the values of λ_i, $i=0,1$ for $f_{\bar{y}_0}$, $f_{\bar{y}_1}$ and $f_{\bar{y}_2}$ computed earlier we find

$$\begin{aligned} C_0 &= \begin{bmatrix} s_0+1/2-p & 1 \\ \frac{p-s_0}{2}+p(s_0-p) & p \end{bmatrix}, \quad C_1 = \begin{bmatrix} s_1+1/2-q & 0 \\ \frac{1-s_1-p}{2}+p(s_1-q) & 1 \end{bmatrix}, \\ C_2 &= \begin{bmatrix} 0 & 0 \\ \frac{1-s_0-q}{2}+q(s_0-p) & q \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0 & 0 \\ \frac{q-s_1}{2}+q(s_1-q) & 0 \end{bmatrix}. \end{aligned} \tag{3.8}$$

For later use we define the inner product

$$\langle \Phi,\Phi\rangle = \int_{\mathbb{R}} \Phi\Phi^* dx = \int_{\mathbb{R}} \begin{bmatrix} \phi^0(x)\phi^0(x) & \phi^0(x)\phi^1(x) \\ \phi^1(x)\phi^0(x) & \phi^1(x)\phi^1(x) \end{bmatrix} dx = E^2,$$

where $E^2 = \begin{bmatrix} \|\phi^0\|^2 & 0 \\ 0 & \|\phi^1\|^2 \end{bmatrix}$.

We will now show that even if longer supports are considered compactly supported, continuous orthogonal scaling functions can only be constructed for those values (s_0,s_1) for which $p(s_0,s_1)=0$ with $|s_0|<1$ and $|s_1|<1$.

Lemma 3.4. *Suppose for a given three dimensional subspace V of $C(I)$, there is no orthonormal basis with two functions vanishing at one endpoint and the remaining function vanishing at the other endpoint. Then any continuous, compactly supported pair of functions ϕ^1 and ϕ^2 composed of linear combinations of the basis elements of V and their integer translates constructed so that $\langle \phi^i(x), \phi^j(x-k)\rangle = \delta_{i,j}\delta_{k,0}$, must have the property that the leftmost nonzero components of ϕ^1, and ϕ^2 are linearly dependent as are their rightmost nonzero components.*

Proof. Suppose the support of ϕ^1 and ϕ^2 are $[0,N]$ and $[0,M]$ respectively with $N+M\geq 3$. Because of continuity $\phi^1(0),\phi^2(0),\phi^1(x+N-1)|_{x=1}$, and $\phi^2(x+M-1)|_{x=1}$ all vanish, furthermore $\phi^1(x)|_{[0,1]}$ and $\phi^2(x)|_{[0,1]}$ are orthogonal to $\phi^1(x+N-1)|_{[0,1]}$ and $\phi^2(x+M-1)|_{[0,1]}$. The result now follows since F spans V. $\square$

We note that by rotation it can always be arranged so that $M\neq N$.

Lemma 3.5. *If the hypotheses of the above lemma are satisfied, then no such pair of functions ϕ^1,ϕ^2 exists.*

Proof. Suppose that ϕ^1 and ϕ^2 exist, that $M < N$, and set $y = \frac{\phi^1}{||\phi^1||_2}$, and $z = \frac{\phi^2}{||\phi^2||_2}$. Then $y$and z are orthonormal and for a suitable basis can be represented as

$$\begin{aligned} y &= (y_1, y_2, \ldots, y_M, 0, 0, \ldots, 0) \\ &= ((y_{1,1}, 0, 0), (y_{2,1}, y_{2,2}, y_{2,3}), \ldots, (0, 0, y_{M,3}), (0, 0, 0), \ldots, (0, 0, 0)) \end{aligned}$$

and

$$z = (z_1, z_2, \ldots, z_N) = ((z_{1,1}, 0, 0), (z_{2,1}, z_{2,2}, z_{2,3}), \ldots, (0, 0, z_{N,3}))$$

where $2 \leq M < N$ and $z_{1,1} > 0$.

Consider the rotation r defined on pairs of vectors of the above form

$$r(y, z) = \frac{1}{\sqrt{y_{1,1}^2 + z_{1,1}^2}}(z_{1,1}y - y_{1,1}z, y_{1,1}y + z_{1,1}z) = (\tilde{y}, \tilde{z}),$$

where

$$\begin{aligned} \tilde{y} &= ((0, 0, 0), (\tilde{y}_{2,1}, \tilde{y}_{2,2}, \tilde{y}_{2,3}), \ldots, (\tilde{y}_{N,1}, \tilde{y}_{N,2}, \tilde{y}_{N,3})) \\ \tilde{z} &= ((\sqrt{z_{1,1}^2 + y_{1,1}^2}, 0, 0), (\tilde{z}_{2,1}, \tilde{z}_{2,2}, \tilde{z}_{2,3}),), \ldots, (\tilde{z}_{N,1}, \tilde{z}_{N,2}, \tilde{z}_{N,3})), \end{aligned}$$

and the shift map s defined on the range of r by

$$s(\tilde{y}, \tilde{z}) = ((\tilde{y}_2, \tilde{y}_3, \ldots, \tilde{y}_N, 0), (\tilde{z}_1, \ldots, \tilde{z}_N)) = (\hat{y}, \hat{z}).$$

Both s and r are continuous, and $\|\hat{z}\|_2 = \|\hat{y}\|_2 = 1$, and $\hat{y}$ and $\hat{z}$ satisfy the necessary orthogonality relations. The operations also preserve continuity of the components, so the functions corresponding to $\hat{y}$ and $\hat{z}$ are continuous. By Lemma 3.4, we know that $\hat{y}_1$ and $\hat{z}_1$ are linearly dependent, so $\hat{y}_{1,2} = 0 = \hat{y}_{1,3}$. It follows that $(\hat{y}, \hat{z})$ is back in the domain of r, so we can iterate the map $s \circ r$ on (y, z) to produce a sequence $(s \circ r)^j(y, z) = (y^{(j)}, z^{(j)})$ for $j = 0, 1, 2, \ldots$. Since this sequence is contained in a compact set we can extract a convergent subsequence with limit say (Y, Z). By continuity of the inner product, we have $\|Y\|_2 = \|Z\|_2 = 1$, Y and Z satisfy the orthogonality relations, and the functions corresponding to Y and Z are continuous.

Observe that $z_{1,1}^{(j)}$ is monotone increasing in j. Since it is a component of a unit vector, it is bounded above by 1, and hence converges to some number, $Z_{1,1}$. We also have that $(z_{1,1}^{(j)})^2$ converges (to $Z_{1,1}^2$), so that the increments of this sequence, $(z_{1,1}^{(j+1)})^2 -$

$(z_{1,1}^{(j)})^2 = (y_{1,1}^{(j)})^2$, must converge to zero. We claim that for each $i \in \{1, 2, \ldots, N-1\}$, $\lim_{j\to\infty} y_{i,1}^{(j)} = 0$. We have just shown that this is true for $i = 1$, so suppose it is true for some i. Then for each j, we have

$$|y_{i,1}^{(j+1)}| = \left| \frac{z_{1,1}^{(j)} y_{i+1,1}^{(j)} - y_{1,1}^{(j)} z_{i+1,1}^{(j)}}{\sqrt{(y_{1,1}^{(j)})^2 + (z_{1,1}^{(j)})^2}} \right|.$$

Since $y_{1,1}^{(j)}$ and $z_{1,1}^{(j)}$ are both components of unit vectors, each is no larger than 1. Thus the denominator (above) is no larger than $\sqrt{2} < 2$, and we have

$$|y_{i,1}^{(j+1)}| \geq \tfrac{1}{2}|z_{1,1}^{(j)} y_{i+1,1}^{(j)}| - \tfrac{1}{2}|y_{1,1}^{(j)} z_{i+1,1}^{(j)}|,$$

or

$$\begin{aligned} |z_{1,1}^{(j)} y_{i+1,1}^{(j)}| &\leq 2|y_{i,1}^{(j+1)}| + |y_{1,1}^{(j)} z_{i+1,1}^{(j)}| \\ &\leq 2|y_{i,1}^{(j+1)}| + |y_{1,1}^{(j)}|. \end{aligned}$$

By our induction hypothesis, this last expression converges to zero. Furthermore, $z_{1,1}^{(j)} \geq z_{1,1} > 0$, so $\lim_{k\to\infty} y_{i+1,1}^{(j)} = 0$, and the claim is established by induction.

Now, our limit point (Y, Z) must have the property that $Y_{i,1} = 0$, for $i = 1, 2, \ldots, N$. But by the lemma, we also know that the leftmost nonzero 3-tuple in Y, say Y_p, is linearly dependent on $Z_1 = (Z_{1,1}, 0, 0)$. So, Y_p has the form $(Y_{p,1}, 0, 0)$, and we have just shown that $Y_{p,1} = 0$. Therefore Y must be zero, which contradicts the fact that $\|Y\| = 1$. $\square$

With the above lemmas we are now able to prove,

Theorem 3.6. *If s_0 and s_1 do not satisfy the relation $p(s_0, s_1) = 0$ (equation (3.6)) then there are no continuous, compactly supported, orthogonal scaling functions formed from the AFIF generated by s_0 and s_1 such that the L^2 closure of the linear span these functions and their integer translates is $\tilde{V}_0$.*

Proof. It is easy to show that $\tilde{V}_0$ cannot be spanned by the integer translates of one continuous compactly supported scaling function. This is because if the scaling function is supported on $[0, M], M > 1$ and we consider an interval of length N there will be at most $N + M - 1$ translates of ϕ supported on this interval. However the same interval will

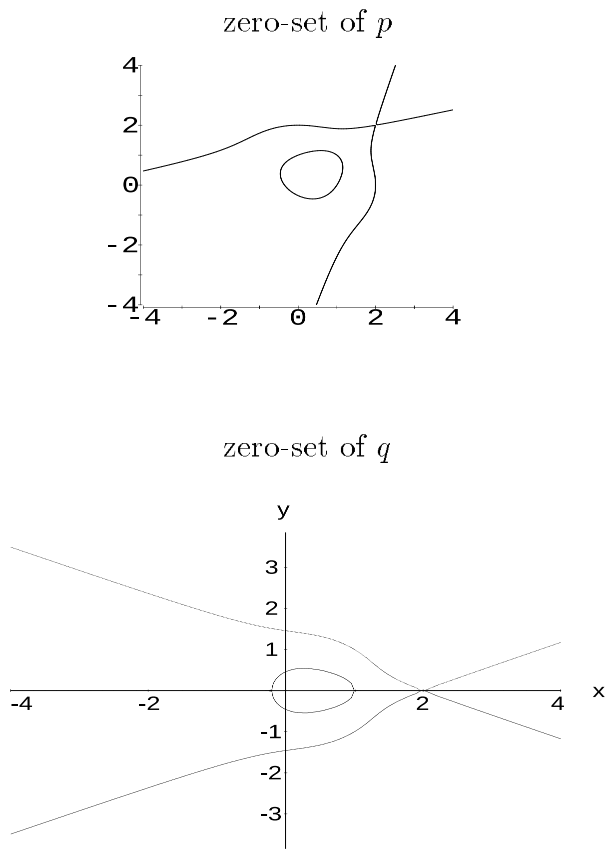


Figure 1

contain $2N+1$ interpolation points. Consequently for large enough N there will not be enough translates of ϕ to match all the necessary conditions. By Lemma 3.5 we need now only consider pairs s_0 and s_1 that allow an orthogonal basis for S_0 in which at least two basis vectors vanish at either zero or one. But Lemma 3.1 and Corollary 3.2 show that this can occur only in the case when $p(s_0, s_1) = 0$, which proves the result. $\square$

Since compactly supported continuous scaling functions constructed from AFIF occur only for the pairs (s_0, s_1), $|s_0| < 1$, $|s_1| < 1$, such that $p(s_0, s_1) = 0$, we examine the zero-set of this polynomial. The next lemma shows that the particular set we are interested in is convex. This confirms the contour plot given in Figure 1.

Lemma 3.7. *The zero-set of the polynomial*

$$p(s_0, s_1) = 2s_0^4 + 6s_0^3 - 7s_0^3 s_1 + 18s_0^2 s_1 - 28s_0^2 - 7s_0 s_1^3 + 18s_0 s_1^2 - 14s_0 s_1 + 12s_0 + 2s_1^4 + 6s_1^3 - 28s_1^2 + 12s_1 + 8,$$

has two connected components, one a convex closed curve and the other a pair of asymptotically linear curves that cross at a point.

Proof. First, we transform the polynomial using the change of variables

$$s_0 = x - y$$

$$s_1 = x + y,$$

to get a new polynomial,

$$q(x, y) = 18y^4 + (-42 + 24x^2)y^2 - 10x^4 + 48x^3 - 70x^2 + 24x + 8.$$

Note that q is symmetric with respect to y for all values of x. The transformation is just a rotation by 45° and a dilation by $\sqrt{2}$, so it preserves all relevant properties of the zero-set.

Now, we want to find out where the transformed polynomial, q, takes the value zero. To do this we consider the equation $q(x, y) = 0$ and solve for y as a function of x. This gives

$$y = \pm\sqrt{\tfrac{7}{6} - \tfrac{2}{3}x^2 + \tfrac{1}{6}\sqrt{3}\sqrt{12x^4 - 32x^3 + 28x^2 - 16x + 11}},$$

or

$$y = \pm\sqrt{\tfrac{7}{6} - \tfrac{2}{3}x^2 - \tfrac{1}{6}\sqrt{3}\sqrt{12x^4 - 32x^3 + 28x^2 - 16x + 11}}.$$

The first solution gives a pair of asymptotically linear curves that cross at $x = 2$, $y = 0$. The second solution, which is real-valued only for $x \in [-\frac{1}{5}, 1]$, gives two halves of a symmetric closed curve, γ, and is the one that we are primarily interested in.

Let f denote the positive branch of γ. If f is a concave down function, it follows by symmetry that γ is a convex curve. Furthermore, since the square root function is monotone concave down, it suffices to show that f^2 $(= f \cdot f)$ is concave down. Thus, we wish to demonstrate that the second derivative of f^2 is nonpositive on $[-\frac{1}{5}, 1]$. That is,

$$0 \geq -\tfrac{4}{3} + \frac{\sqrt{3}(48x^3 - 96x^2 + 56x - 16)^2}{24p^{3/2}} - \frac{\sqrt{3}(144x^2 - 192x + 56)}{12p^{1/2}},$$

where $p = 12x^4 - 32x^3 + 28x^2 - 16x + 11$. Note that p is positive for all values of x, so the above expression is well-defined. Also, since the denominators are powers of p, they are positive as well, and we can multiply them out to get an equivalent inequality,

$$0 \geq -4p^{3/2} - \sqrt{3}(-144x^6 + 576x^5 - 888x^4 + 832x^3 - 780x^2 + 528x - 122).$$

To show this, we approximate $\sqrt{p}$ with a linear polynomial, $q = \sqrt{3}(\frac{7}{4} - \frac{3}{4}x)$. Observe that $q \leq \sqrt{p}$ on $[-\frac{1}{5}, 1]$, since $p - q^2 = \frac{1}{16}(x-1)(192x^3 - 320x^2 + 101x - 29)$ is nonnegative on that interval. Thus, by substituting pq for $p^{3/2}$ in the above and expanding, we have

$$\begin{aligned}
&-4p^{3/2} - \sqrt{3}(-144x^6 + 576x^5 - 888x^4 + 832x^3 - 780x^2 + 528x - 122) \\
&\quad \leq \sqrt{3}(-144x^6 + 612x^5 - 1068x^4 + 1140x^3 - 1024x^2 + 673x - 199) \\
&\quad \leq \sqrt{3}(-144x^6 + 612x^5 - 1068x^4 + 1140x^3 - 1024x^2 + 673x - 199) \\
&\qquad + \tfrac{\sqrt{3}}{3125}(37683 - 19989x) \\
&\quad = -\tfrac{4\sqrt{3}}{3125}(5x-4)^2(4500x^4 - 11925x^3 + 11415x^2 - 9729x + 9128) \\
&\quad \leq 0.
\end{aligned}$$

The last inequality is justified, since both factors involving x are nonnegative for all values of x. ☐

Let (a, b) be the appropriate pair that solves the equations $dp/ds_0 = 0$ and $p = 0$, which yields $a \approx -.4628$. Then by examining $p(s_0, s_1)$ and using Lemma 3.7 we see that for every $s_0 \in [a, 1)$ there is at least one and at most two values of s_1, $|s_1| < 1$ with $p(s_0, s_1) = 0$. If $s_0 = s_1$ then $x = 0$ in the polynomial $q(x, y)$ and we find $q(0, y) = -10y^4 + 48y^3 - 70y^2 + 24y + 8 = -2(5y+1)(y-1)(-2+y)^2$. Thus a solution of this equation is $y = -1/5 = s_0 = s_1$ and equations (3.4) and (3.5) give $p_1 = q_1 = -3/10$. Consequently for $s_0 = s_1 = -1/5$ the functions $f_{\bar{y}_0}, f_{\bar{y}_1}$, and $f_{\bar{y}_2}\}$ with $\bar{y}_0 = [1, -3/10, 0]$, $\bar{y}_1 = [0, 1, 0]$, and $\bar{y}_2 = [0, -3/10, 1]$ are mutually orthogonal. The scaling functions ϕ^0 and ϕ^1 given as in Theorem 3.3 are shown in Figure 2. Note that ϕ^0 is symmetric about $1/2$ while ϕ^1 is symmetric about 1. Consequently individually both exhibit linear phase [4].

If we normalize $\hat{\Phi}$ so that $\langle \hat{\Phi}, \hat{\Phi} \rangle = I$ then the coefficients in (3.7) can be re-expressed as $\hat{\Phi}(x) = \sum_{i=0}^{3} \widehat{C}_i \sqrt{2} \hat{\Phi}(2x - i)$ where $\widehat{C}_i = \frac{E^{-1} C_i E}{\sqrt{2}}$. The $\{\widehat{C}_i\}$ for $s_0 = s_1 = -1/5$ are

$$\widehat{C}_0 = \begin{pmatrix} 3\sqrt{2}/10 & 4/5 \\ -1/20 & -3\sqrt{2}/20 \end{pmatrix}, \qquad \widehat{C}_1 = \begin{pmatrix} 3\sqrt{2}/10 & 0 \\ 9/20 & \sqrt{2}/2 \end{pmatrix},$$

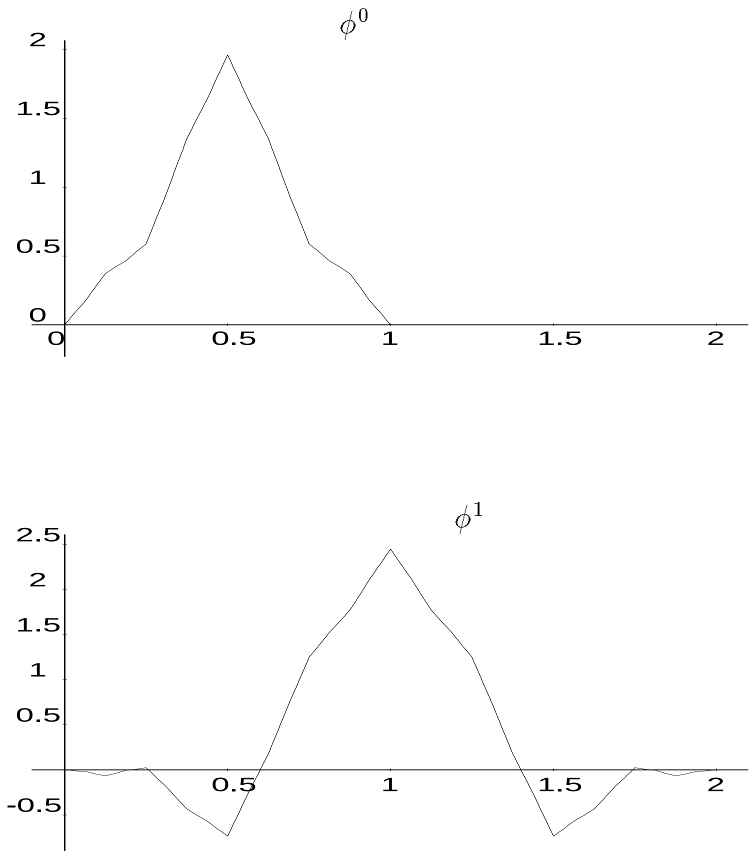


Figure 2

$$\widehat{C}_2 = \begin{pmatrix} 0 & 0 \\ 9/20 & -3\sqrt{2}/20 \end{pmatrix}, \qquad \widehat{C}_3 = \begin{pmatrix} 0 & 0 \\ -1/20 & 0 \end{pmatrix}.$$

The values of p_1 and q_1 for other acceptable pairs s_0, s_1 are given in Table 1.

We now consider the smoothness and approximation order of the function ϕ^i, $i = 0, 1$. Recall that the Hölder exponent of $f \in C(I)$ at x is

$$\alpha_x = \lim_{\epsilon \longrightarrow 0} \inf\{\log|f(x) - f(y)|/\log|x - y| : y \in B(x, \epsilon)\}$$

and $\alpha = \inf\{\alpha_x, x \in I\}$ is called the Hölder exponent of f.

Let S be a closed subspace of $L^2(\mathbb{R})$, $E(f, S) = \min\{\|f - s\| : s \in S\}$, $\hat{f}$ be the Fourier transform of f, and $w_2^k(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \|f\|_{w_2^k} = \|(1 + |\cdot|^k)\hat{f}\| < \infty\}$. Following de Boor, DeVore, and Ron [6] we say that S provides approximation order k, if for every

$f \in w_2^k(\mathbb{R})$,

$$E(f, S^h) \leq C_S h^k \|f\|_{w_2^k(\mathbb{R})}$$

where $S^h = \{s(\frac{\cdot}{h}) : s \in S\}$.

Lemma 3.8. *Given any pair (s_0, s_1), $|s_0| < 1$, $|s_1| < 1$, let $f \in S_0$. If $|s_0| < 1/2$ and $|s_1| < 1/2$ then f is Lipschitz continuous, i.e., there exists an $M < \infty$ such that $\forall\, x, y \in [0,1], |f(x) - f(y)| \leq M|x-y|$. If $\max|s_i| > 1/2$ and f is not a line then f has Hölder exponent $\alpha = -\log\max|s_i|/\log 2$.*

We note that the second part of the lemma has already been proved by Bedford [3] and we include the proof for the convenience of the reader.

Proof. Let $s = \max_i|s_i|$, $\mathbf{i}_n = \{i_1, i_2, \ldots, i_n\}$, $i_j \in \{0,1\}$, $j = 1, \ldots, n$, and $a(\mathbf{i}_n) = i_1/2 + i_2/4 + \cdots + i_n/2^n$. Then for $x \in [a(\mathbf{i}_n), a(\mathbf{i}_n) + 1/2^n]$ we find from (2.3) that

$$\begin{aligned} f(x) = &\sum_{k=1}^{n}\left(2^{k-1}x + b_{i_k} - \sum_{m=1}^{k}\frac{i_m}{2^{m-1}}\right)\prod_{j=1}^{k-1}s_{i_j} \\ &+ \left(\prod_{j=1}^{n}s_{i_j}\right) f\left(2^n x - \sum_{m=1}^{n}2^{n-m}i_m\right). \end{aligned} \tag{3.9}$$

Suppose $1/2^{n+1} \leq |x-y| \leq 1/2^n$ and $x, y \in [a(\mathbf{i}_n), a(\mathbf{i}_n) + 1/2^n]$, then the above formula shows us that

$$|f(x) - f(y)| \leq \sum_{k=1}^{n}(2s)^{k-1}|x-y| + s^n C, \tag{3.10}$$

where $C = 2\|f\|_\infty$. Suppose first that $s < 1/2$ then

$$|f(x) - f(y)| \leq \sum_{k=1}^{n}(2s)^{k-1}|x-y| + (2s)^n 2C|x-y|,$$

where the fact that $1 \leq 2^{n+1}|x-y|$ has been used to obtain the last term in the above expression. Since $2s < 1$ we find that

$$|f(x) - f(y)| \leq \frac{1+2C}{1-2s}|x-y|. \tag{3.11}$$

If $1/2^{n+1} \le |x-y| \le 1/2^n$ but x and y are not in the same dyadic interval let x_0 be the boundary point between the respective intervals x and y are located in. Then $|f(x)-f(y)| \le |f(x)-f(x_0)| + |f(x_0)-f(y)|$ and applying (3.11) gives

$$|f(x)-f(y)| \le \frac{2(1+2C)}{1-2s}\,|x-y|$$

for all $x, y \in [0,1]$. This gives the Lipschitz continuity of f.

If $s = \max|s_i| > 1/2$ assume without loss of generality that $f(0) = 0 = f(1)$. For if that is not the case we can modify f by adding linear functions $L_1(x)$ and $L_2(x)$ so that $\hat{f} = f + L_1 + L_2$ is still in S_0, $\hat{f}(0) = \hat{f}(1) = 0$. If $1/2^{n+1} \le |x-y| \le 1/2^n$, $a(\mathbf{i}_n)$, $x, y \in [a(\mathbf{i}_n), a(\mathbf{i}_n) + 1/2^n]$ (3.10) implies

$$|f(x)-f(y)| \le s^n \left(C + \sum_{k=1}^{n} (2s)^{k-n-1} \right) = s^n \left(C + \frac{2s}{1-\frac{1}{2s}} \right).$$

Since $1/2^{n+1} \le |x-y| \le 1/2^n$ we find that $|x-y|^\alpha \ge 2^{-\alpha(n+1)} = 2^{-\alpha}2^{-\alpha n} = 2^{-\alpha}s^n$ with $\alpha = \frac{\ln s}{\ln 1/2}$. Hence

$$|f(x)-f(y)| \le 2^\alpha \left(C + \frac{2s}{1-\frac{1}{2s}} \right) |x-y|^\alpha \tag{3.12}$$

for x, y in the same dyadic interval of length $1/2^n$. If x, y are not in the same dyadic interval then using the same argument as in the case when $2s < 1$ we find

$$|f(x)-f(y)| \le 2^{\alpha+1} \left(C + \frac{2s}{1-\frac{1}{2s}} \right) |x-y|^\alpha$$

for all $x, y \in [0,1]$. We need only show that α is the largest possible exponent. Suppose without loss of generality that $s = |s_0|$. Since f vanishes at 0 and 1 and $f(\frac{1}{2}) \neq 0$ (since f is not a line), there exist distinct points x_0 and $y_0 \in [0,1]$ such that $f(x_0) \neq f(y_0)$ and $\frac{x_0-y_0}{f(x_0)-f(y_0)} > 0$. With $x_n = x_0/2^n$ and $y_n = y_0/2^n$ we find from (3.9) that

$$f(x_n) - f(y_n) = \sum_{k=1}^{n-1} (2s_0)^{k-1} \frac{x_0-y_0}{2^n} + s_0^n (f(x_0) - f(y_0)).$$

Therefore

$$\begin{aligned} |f(x_n) - f(y_n)| &= s^n |f(x_0) - f(y_0)| \left| 1 + \frac{x_0-y_0}{f(x_0)-f(y_0)} \left(\frac{1}{2s}\right)^2 \frac{1-(\frac{1}{2s_0})^{n-2}}{1-\frac{1}{2s_0}} \right| \\ &\ge s^n C \ge k|x_n - y_n|^\alpha \end{aligned}$$

which proves the result. $\square$

We can now prove

Theorem 3.9. *Suppose the pair* (s_0, s_1) *is such that* $|s_0| < 1$ *and* $|s_1| < 1$ *with* p_1 *and* q_1 *satisfying (3.4) and (3.5) respectively. Then* V_0 *provides approximation order 2. If* s_0 *and* s_1 *are both in magnitude less than 1/2 then* ϕ^i, $i = 0, 1$ *are both Lipschitz. If* $s = \max|s_i| > 1/2$ *then* ϕ^i, $i = 0, 1$ *have Hölder exponent* $\alpha = -\log s / \log 2$.

Proof. In order to show that V_0 provides approximation order 2. We need only prove that the hat function

$$g(x) = \begin{cases} x & 0 \le x \le 1, \\ 2 - x & 1 \le x \le 2 \end{cases}$$

is in V_0 [20]. It is easy to see from (2.3) that for any pair s_0, s_1 with $0 \le |s_0| < 1$ and $0 \le |s_1| < 1$, $f_{[0,1/2,1]}(x) = x$. Consequently,

$$g(x) = (1/2 - p_1)\phi^0(x) + \phi^1(x) + (1/2 - q_1)\phi^0(x - 1).$$

The fact that ϕ^0 and ϕ^1 are Lipschitz when $s < 1/2$ follows from Lemma 3.7. Furthermore Lemma 3.7 also shows that if $s > 1/2$ then the Hölder exponent of ϕ^0 is $\log|s|/\log 1/2$. This will also be true of ϕ^1 once it is shown that ϕ^1 is not piecewise linear. For $s > 1/2$ this can happen only if $p_1 = q_1 = 1/2$, however from (3.1) we find with $p_1 = 1/2$,

$$\int f_{\bar{y}_1} f_{\bar{y}_2} dx = \frac{-2 + s_0}{(-2 + s_0 + s_1)(-4 + s_0 + s_1)} \neq 0$$

which is a contradiction. Therefore $f_{\bar{y}_2}$ is not a line. A similar argument shows that $f_{\bar{y}_0}$ cannot be a line and the result now follows. $\square$

We complete this section by computing the Fourier transform of ϕ^0 and ϕ^1. To this end set $g(x) = e^{ikx}$, $N = 2$ in (2.6) to find

$$\begin{aligned} \hat{f}_{\bar{y}}(k) = \int_0^1 e^{ikx} f_{\bar{y}}(x) dx = \frac{1}{2} \int_0^1 (a_0 x + b_0) e^{ikx/2} dx \\ + 1/2 \int_0^1 e^{ik(x+1)/2} (a_1 x + b_1) dx \\ + 1/2(s_0 + e^{ik/2} s_1) \hat{f}_{\bar{y}}(k/2). \end{aligned}$$

For ϕ^1, $a_0 = 1$, $b_0 = 0$, $a_1 = -1$ and $b_1 = 1$. Consequently,

$$\hat{\phi}^0(k) = 8e^{ik/2} \frac{\sin^2 k/4}{k^2} + \frac{1}{2}(s_0 + e^{ik/2} s_1)\hat{\phi}^0(k/2).$$

With $C(k) = \frac{1}{2}(s_0 + e^{ik/2}s_1)$ and $h_1(k) = 8e^{ik/2}\frac{\sin^2 k/4}{k^2}$ we find that

$$\hat{\phi}^0(k) = \sum_{n=0}^{\infty}\left(\prod_{j=1}^{n} C(k/2^{j-1})\right) h_1(k/2^n).$$

The above series converges uniformly for $k \in \mathbb{R}$ since $|C(k/2^j)| \leq \frac{|s_0|+|s_1|}{2} = r < 1$, $j = 1, 2\ldots$ and since $|\sin x/x| \leq 1$ for all $x \in \mathbb{R}$.

To compute the Fourier transform of ϕ^1 we use the fact that ϕ^1 can be written in terms of the hat function g and ϕ^0 as was shown above. Thus

$$\hat{\phi}^1(k) = \hat{g}(k) - (1/2 - p_1 - (1/2 - q_1)e^{ik})\hat{\phi}^0(k).$$

Below the Fourier transforms of ϕ^0 and ϕ^1 are plotted when $s_0 = s_1 = -1/5$.

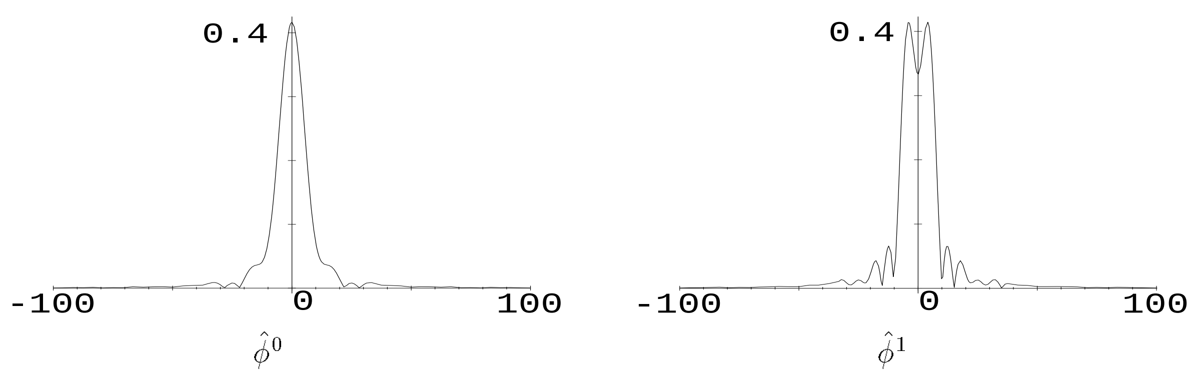


Figure 3

IV. Construction of Compactly Supported Wavelets

Having constructed a multiresolution analysis based on the vector $\phi = \binom{\phi^0}{\phi^1}$ and its dilates and translates we now construct compactly supported, continuous wavelets. If $\tilde{W}_0$ is the orthogonal complement of $\tilde{V}_0$ in $\tilde{V}_1$ then any $\Psi \in \tilde{W}_0$ can be written as $\Psi(x) = \binom{\psi^0(x)}{\psi^1(x)}$ where $\psi^i(x)|_{[j/2,(j+1)/2)}$, $i = 0, 1$ is an AFIF with interpolation points at the quarter integers. Since $\Phi(x) = \binom{\phi^0(x)}{\phi^1(x)}$ is supported on $[0, 2]$, we shall look for $\Psi(x)$ to be supported on the same interval. Therefore set

$$\psi^i(x) = \begin{cases} g_{\bar{y}_1^i}(x), & 0 \le x \le 1/2, \\ g_{\bar{y}_2^i}(x), & 1/2 \le x \le 1, \\ g_{\bar{y}_3^i}(x), & 1 \le x \le 3/2, \\ g_{\bar{y}_4^i}(x), & 3/2 \le x \le 1, \end{cases} \qquad i = 0, 1. \tag{4.1}$$

Here $\bar{y}_j^i$ are the values $\psi^i(x)$ taken at the quarter integer points in the intervals indicated. In order to preserve continuity and keep the support of $\Phi(x)$ on $[0,2]$, $\bar{y}_1^i = [0, a_1^i, a_2^i]$, $\bar{y}_2^i = [a_2^i, a_3^i, a_4^i]$, $y_3^i = [a_4^i, a_5^i, a_6^i]$ and $\bar{y}_4^i = [a_6^i, a_7^i, 0]$, $i = 0, 1$. The coefficients a_j^i, $j = 1, 2, \ldots, 7$, $i = 0, 1$ are adjusted so that $\Psi(x)$ is orthogonal to $\Phi(x)$ and its translates and $\langle \psi_0^0(x), \psi^1(x) \rangle = 0$. In the case when $s_0 = -1/5 = s_1$ and $p_1 = -3/10 = q_1$ it follows from (2.7), (2.8), and (2.9) that for $i = 0$ or 1,

$$\int_0^2 \psi^i(x)\phi^0(x)dx = \frac{5}{32}a_1^i + \frac{41}{96}a_2^i + \frac{5}{32}a_3^i + \frac{3}{64}a_4^i, \tag{4.2}$$

$$\int_0^1 \psi^i(x)\phi^0(x+1)dx = \frac{3}{64}a_4^i + \frac{5}{32}a_5^i + \frac{41}{96}a_6^i + \frac{5}{32}a_7^i, \tag{4.3}$$

$$\int_0^2 \psi^i(x)\phi^1(x)dx = -\frac{1}{48}a_1^i - \frac{1}{20}a_2^i + \frac{3}{16}a_3^i + \frac{107}{240}a_4^i + \frac{3}{16}a_5^i - \frac{1}{20}a_6^i - \frac{1}{48}a_7^i, \tag{4.4}$$

$$\int_0^2 \psi^i(x)\phi^1(x+1)dx = \frac{3}{16}a_1^i - \frac{1}{20}a_2^i - \frac{1}{48}a_3^i - \frac{1}{160}a_4^i, \tag{4.5}$$

$$\int_0^2 \psi^i(x)\phi^1(x-1)dx = -\frac{1}{160}a_4^i - \frac{1}{48}a_5^i - \frac{1}{20}a_6^i + \frac{3}{16}a_7^i, \tag{4.6}$$

and

$$\begin{aligned} \int_0^2 \psi^0(x)\psi^1(x)dx = {} & \frac{25}{96}(a_1^0a_1^1 + a_3^0a_3^1 + a_5^0a_5^1 + a_7^0a_7^1) \\ & + \frac{73}{192}(a_2^0a_2^1 + a_6^0a_6^1 + a_4^0a_4^1 + a_6^0a_6^1) \\ & + \frac{3}{128}(a_4^1a_2^0 + a_4^0a_2^1 + a_6^0a_4^1 + a_6^1a_4^0) \\ & + \frac{5}{64}(a_2^0a_1^1 + a_1^0a_2^1 + a_3^0a_2^1 + a_2^0a_3^1 \\ & + a_4^0a_3^1 + a_3^0a_4^1 + a_5^0a_4^1 + a_4^0a_5^1 + a_6^0a_5^1 \\ & + a_5^0a_6^1 + a_7^0a_6^1 + a_6^0a_7^1). \end{aligned} \tag{4.7}$$

The above equations once set equal to zero will fix all but three of the unknowns. Two of these are used to normalize the integrals of ψ^0 and ψ^1. One unknown remains because of the one parameter family of rotations taking Ψ into other mother wavelets. A remarkable fact to be shown below is that once (4.3), (4.5) and (4.6) are satisfied then $\langle \Psi, \Psi(\cdot + i)\rangle = 0$ $\forall\ i \in \mathbb{Z}$, $i \neq 0$. A solution to the above equations that give $\langle \Psi, \Psi\rangle = I$ with Ψ having the smallest possible support is

$$
\begin{aligned}
y_1^0 &= \left[0, \frac{-3\sqrt{2}}{200}, \frac{9\sqrt{2}}{20}\right], \qquad & y_2^0 &= \left[\frac{9\sqrt{2}}{20}, \frac{-273\sqrt{2}}{200}, \frac{\sqrt{2}}{2}\right], \\
y_3^0 &= \left[\frac{\sqrt{2}}{2}, \frac{5\sqrt{2}}{200}, \frac{-3\sqrt{2}}{20}\right], \qquad & y_4^0 &= \left[\frac{-3\sqrt{2}}{20}, \frac{\sqrt{2}}{200}, 0\right],
\end{aligned}
\tag{4.8}
$$

$$
\begin{aligned}
y_1^1 &= [0, 0, 0]\,, \qquad & y_2^1 &= \left[0, \frac{3}{10}, -1\right], \\
y_3^1 &= \left[-1, \frac{48}{25}, \frac{-3}{5}\right], \qquad & y_4^1 &= \left[\frac{-3}{5}, \frac{1}{50}, 0\right],
\end{aligned}
\tag{4.9}
$$

Since $\Psi \in \tilde{V}_1$ we have

$$\Psi(x) = \sum_{i=0}^{3} D_i \Phi(2x - i), \tag{4.10}$$

where for the particular Ψ given by (4.8) and (4.9)

$$
\begin{aligned}
D_0 &= \begin{pmatrix} \sqrt{2}/20 & 3\sqrt{6}/20 \\ 0 & 0 \end{pmatrix}, \qquad & D_1 &= \begin{pmatrix} -9\sqrt{3}/20 & 1/\sqrt{6} \\ 0 & -1/\sqrt{3} \end{pmatrix}, \\
D_2 &= \begin{pmatrix} 3\sqrt{3}/20 & -\sqrt{6}/20 \\ 3\sqrt{6}/10 & -\sqrt{3}/5 \end{pmatrix}, \qquad & D_3 &= \begin{pmatrix} -\sqrt{3}/60 & 0 \\ -\sqrt{6}/30 & 0 \end{pmatrix}.
\end{aligned}
\tag{4.11}
$$

In order to see why the orthogonality of Ψ to the translates of Φ implies the orthogonality of Ψ to its non zero integer translates we examine the multiresolution analysis arising from Φ and consider the slightly more general case where $\Phi \in \mathbb{R}^N$ which we will use in section V. In this case each C_i in (3.7) is an $N \times N$ matrix and Φ satisfies the N-scale dilation equation

$$\Phi(x) = \sum_{i=0}^{2N-1} C_i \Phi(Nx - i).$$

Likewise the mother wavelet Ψ satisfies

$$\Psi(x) = \sum_{i=0}^{2N-1} D_i \Phi(Nx - i),$$

where the matrices D_i are $N(N-1)\times N$ matrices. The orthogonality of $\Phi(x)$ to its integer translates can be re-expressed as,

$$\int \Phi(x)\Phi^*(x-i)dx = \delta_{i,0} I_N = \sum_{k=0}^{2N-1} C_k C^*_{k-Ni}, \qquad \forall\, i \in \mathbb{R}, \tag{4.12}$$

where I_N is the $N \times N$ identity matrix. Likewise the orthogonality of Ψ against its translates gives

$$\sum_{k=0}^{2N-1} D_k D^*_{k-Ni} = \delta_{i,0} I_{N(N-1)}, \qquad \forall\, i \in \mathbb{R}. \tag{4.13}$$

That W_0 is orthogonal to V_0 in V_1 means

$$\sum_{k=0}^{2N-1} C_k D^*_{k-Ni} = 0, \qquad \forall\, i \in \mathbb{R}, \tag{4.14}$$

while $V_1 = V_0 \oplus W_0$ can be expressed as [5]

$$\sum_k C^*_{m-Nk} C_{n-Nk} + D^*_{m-Nk} D_{n-Nk} = \delta_{m,n} I_N \qquad \forall\, n, m \in \mathbb{Z}. \tag{4.15}$$

If we set $H_1 = (C_0, C_1, \ldots, C_{N-1})$, $H_2 = (C_N, C_{N+1}, \ldots, C_{2N-1})$, $G_1 = (D_0, D_1, \ldots, D_{N-1})$, and $G_2 = (D_N, D_{N+1}, \ldots, D_{2N-1})$. Then (4.12) can be recast as

$$H_1 H_2^* = 0 \tag{4.16}$$

and

$$H_1 H_1^* + H_2 H_2^* = I_N \tag{4.17}$$

Likewise (4.13) and (4.14) become

$$G_1 G_2^* = 0 \tag{4.18}$$

$$G_1 G_1^* + G_2 G_2^* = I_{N(N-1)} \tag{4.19}$$

$$H_2 G_1^* = 0 \tag{4.20}$$

$$H_1 G_2^* = 0 \tag{4.21}$$

and

$$H_1 G_1^* + H_2 G_2^* = 0 \tag{4.22}$$

Equation (4.15) can be recast as

$$H_1^* H_1 + H_2^* H_2 + G_1^* G_1 + G_2^* G_2 = I_{N^2} \tag{4.23}$$

and

$$H_1^* H_2 + G_1^* G_2 = 0. \tag{4.24}$$

The general solution to (4.16) is $H_2^* = P_1 Y$ where $P_1 : \mathbb{R}^{N^2} \to \mathbb{R}^{N^2}$ is an orthogonal projection onto the null space of H_1 and Y is any $N^2 \times N$ matrix. Likewise from (4.21) it is easy to see that $G_2^* = P_1 X$ where X is any $N^2 \times N(N-1)$ matrix. If we set $H_1^* = (I_{N^2} - P_1)Y$ then (4.16) and (4.17) are satisfied and $Y = H_1^* + H_2^*$. Observe that $Y^* Y = I_N$. Equations (4.20), (4.18) and (4.19) now suggest that $G_1^* = (I_{N^2} - P_1)X$ with $X^* X = I_{N(N-1)}$, and (4.21) says that $Y^* X = 0$. Since Y is an $N^2 \times N$ matrix and X is an $N^2 \times N(N-1)$ matrix it is an idea of Gil Strang that X can be obtained by letting its columns be the remaining orthonormal basis vectors for the $\mathbb{R}^{N^2}$ the first N basis vectors being the columns of Y. That is we choose X so that the matrix $(Y\ X)$ is an orthogonal matrix. If this is done (4.23) and (4.24) will also be satisfied since $YY^* + XX^* = I_{N^2}$. Thus equations (4.16) – (4.24) and the assumptions on G_1, G_2, H_1, H_2 can be summarized as

$$\begin{pmatrix} Y^* \\ X^* \end{pmatrix} (Y\ X) = I_{N^2} = (Y\ X) \begin{pmatrix} Y^* \\ X^* \end{pmatrix}.$$

The filters arising from the above equations are closely related to those found in Vetterli [22] (also see Strang & Strela [21]). Thus we have shown

Theorem 4.1. *Let $\{C_i\}_{i=0}^{2N-1}$ be $N \times N$ matrices satisfying* (4.12) *then there exist $N(N-1) \times N$ matrices $\{D_i\}_{i=0}^{2N-1}$ constructed as above so that equations* (4.13), (4.14) *and* (4.15) *are satisfied.*

We now show that for any wavelet Ψ that is constructed using the above scaling function Φ the minimum length of the support of any of its components is $3/2$ and at least one component must have a support greater than or equal to two. Before proving the next lemma we note that for $|s_0| < 1$ and $|s_1| < 1$, and $f \in \tilde{V}_0$,

$$f(x) = \sum_i (c_i^0 \phi^0(x-i) + c_i^1 \phi^1(x-i)), \qquad 4.25a$$

where

$$c_i^1 = f(i+1), \tag{4.25b}$$

and

$$c_i^0 = f(i+\frac{1}{2}) - c_i^1\phi^1(\frac{1}{2}) - c_{i-1}^1\phi^1(\frac{3}{2}). \tag{4.25c}$$

Lemma 4.2. *For any pair* (s_0, s_1) *such that* $p(s_0, s_1) = 0$ *with* $|s_0| < 1$ *and* $|s_1| < 1$ *there is no wavelet function supported on* $[0,1]$ *or on* $[\frac{1}{2}, \frac{3}{2}]$.

Proof. Denote by U_1 the restriction of $\tilde{V}_1$ to the interval $[0,1]$. That is, $U_1 = \{f|_{[0,1]} : f \in \tilde{V}_1\}$. We see that U_1 is a 5-dimensional vector space. For notational simplicity, let

$$\begin{array}{ll} x_1 = \phi^1|_{[0,1]} & x_2 = \phi^1(\cdot+1)|_{[0,1]} \\ x_3 = \phi^1(2\cdot+1)|_{[0,1]} & x_4 = \phi^1(2\cdot-1)|_{[0,1]}. \end{array}$$

Note that $\text{rank}\{\phi^0, x_1, x_2, x_3, x_4\} = 5$. Suppose that ψ^0 is a wavelet supported on $[0,1]$. Then ψ^0 is orthogonal to x_1, x_2, and ϕ^0 because of the orthogonality between wavelets and scaling functions. Also, from (4.25) applied to functions in $\tilde{V}_1$ we see that ψ^0 is orthogonal to $\phi^1(2\cdot+1)$ and $\phi^1(2\cdot-1)$ (and hence to x_3 and x_4) since ψ^0 vanishes at both 0 and 1. Thus $\psi^0 \in \{\phi^0, x_1, x_2, x_3, x_4\}^- = \{0\}$. This cannot be, so there is no such wavelet, ψ^0.

Now, suppose that ψ^0 is supported on $[\frac{1}{2}, \frac{3}{2}]$, and let

$$\begin{array}{ll} y_1 = \phi^0|_{[0,\frac{1}{2}]} & y_2 = \phi^1|_{[0,\frac{1}{2}]} \\ y_3 = \phi^0(\cdot+\frac{1}{2})|_{[0,\frac{1}{2}]} & y_4 = \phi^1(\cdot+\frac{3}{2})|_{[0,\frac{1}{2}]} \\ z_1 = \psi^0(\cdot+\frac{1}{2})|_{[0,\frac{1}{2}]} & z_2 = \psi^0(\cdot+1)|_{[0,\frac{1}{2}]}. \end{array}$$

Note that the above are AFIF's on $[0, \frac{1}{2}]$. Since $f_{\bar{y}_2}(2\cdot)$ is orthogonal to $f_{\bar{y}_1}(2\cdot)$ it follows from (3.8) that if $s_0 \neq 0$ then y_1 and y_2 are linearly independent. A similar argument shows that if $s_1 \neq 0$ then y_3 and y_4 are linearly independent. Orthogonality between ψ^0 and $\phi^i(\cdot - j)$ dictates that z_1 must be orthogonal to y_3 and y_4. This implies that z_1 is orthogonal to all functions in $\tilde{V}_1|_{[0,\frac{1}{2}]}$ vanishing at $\frac{1}{2}$ since this space is three dimensional with basis $\{f_{\bar{y}_i}(2\cdot)\}_{i=0}^3$. Similarly, z_2 is orthogonal to all functions in $\tilde{V}_1|_{[0,\frac{1}{2}]}$ vanishing at 0. Thus it follows from (4.25) that ψ^0 is a multiple of $\phi^1(2\cdot-1)$. Now, ϕ^1 does not vanish at 1, but we require that $\langle\phi^1, \psi^0\rangle = 0$. This cannot be, since among the functions

$\phi^i(2\cdot -j)$, of which ϕ^1 is a linear combination, the only one that does not vanish at 1 is $\phi^1(2\cdot -1)$.

For the case of $s_0 = 0$ (or similarly for $s_1 = 0$), we can apply the quadrature formulas (2.9). Solving $p(0, s_1) = 0$, yields $s_1 = \sqrt{7} - 3$ which implies that y_3 and y_4 linearly independent. Thus z_1 is still a multiple of the left half of $\phi^1(2\cdot)$. Rescale ψ^0 so that $z_1(\frac{1}{2}) = 1$, hence z_2 must have the form $f_{[1,r,0]}(2\cdot)$, where r is yet to be determined. Since $s_0 = 0, \phi^0(x)|_{[0,\frac{1}{2}]} = f_{[0,\frac{1}{2},1]}$ which is a line. From (2.9),

$$\langle \psi^0(\cdot + 1), \phi^0 \rangle = \langle z_2, y_1 \rangle = \frac{2 + 12r - 6s_1 + s_1^2}{6(s_1 - 2)(s_1 - 4)}.$$

In order for this to be equal to zero $r = \frac{s_1}{2} - \frac{s_1^2}{12} - \frac{1}{6} = \sqrt{7} - 3$. We must also have $\langle \psi^0, \phi^1 \rangle = 0$. From the above remarks we find that, $\psi^0|_{[\frac{1}{2},1]} = f_{[0,p,1]}(2\cdot -1)$, while $\phi^1|_{[\frac{1}{2},1]} = f_{[p,\frac{1-s_1+p+2s_1p}{2},1]}(2\cdot -1)$, and $\phi^1|_{[1,\frac{3}{2}]} = f_{[1,\frac{1+q}{2},q]}(2\cdot -2)$, where from (3.5) $q = \frac{3s_1^2+2s_1-2}{4s_1+8} = \sqrt{7} - 3$, and from (3.4) $p = \frac{1-s_1^2}{s_1^2-4} = \frac{\sqrt{7}-4}{6}$. From (2.9) we find after change of variables,

$$\begin{aligned}\langle \psi^0, \phi^1 \rangle &= \frac{1}{2}\int_0^1 f_{[0,p,1]}(x) f_{[p,\frac{1-s_1+p+2s_1p}{2},1]}(x)dx + \frac{1}{2}\int_0^1 f_{[1,r,0]}(x) f_{[1,\frac{1+q}{2},q]}(x)dx \\ &= \frac{(7+5\sqrt{7})(6r - 4\sqrt{7} + 25)}{756}.\end{aligned}$$

Thus in order for the above integral to be equal to zero $r = \frac{4\sqrt{7}-25}{6}$ which cannot be. $\square$

This leads to

Theorem 4.3. *For any pair (s_0, s_1) with $p(s_0, s_1) = 0, |s_0| < 1$, and $|s_1| < 1$ let Ψ be a wavelet. Then the support of one component of Ψ must be of length $\geq 3/2$ while the support of the other component must be of length ≥ 2.*

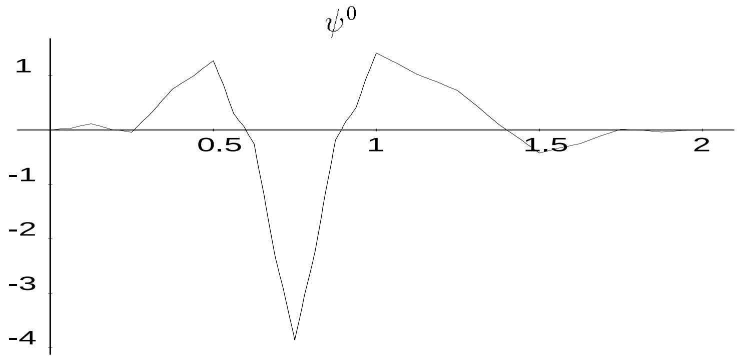
Proof. Lemma 4.2 shows that both components of Ψ must have supports with lengths at least $3/2$. It is easy to see that not both may be supported on $[0, 3/2]$ since in that case the pieces supported on $[1, 3/2]$ or $[0, 1/2]$ must be linearly dependent. A rotation of the components would then allow us to find a wavelet whose support length is 1 which would contradict Lemma 4.2. A similar argument shows that both components of Ψ may not have support $[1/2, 2]$.

Suppose $\Psi = \binom{\psi^1}{\psi^2}$ where supp $\psi^1 = [0, 3/2]$ and supp $\psi^2 = [1/2, 2]$. In this case the matrices in (4.10) would have the form

$$D_0 = \begin{bmatrix} d_1 & d_2 \\ 0 & 0 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} d_3 & d_4 \\ d_5 & d_6 \end{bmatrix},$$
$$D_2 = \begin{bmatrix} d_7 & 0 \\ d_8 & d_9 \end{bmatrix}, \qquad D_3 = \begin{bmatrix} 0 & 0 \\ d_{10} & d_{11} \end{bmatrix}.$$

Set $A = C_0D_2^* + C_1D_3^*$, $B = C_2D_0^* + C_3D_1^*$ and $C = C_0D_0^* + C_1D_1^* + C_2D_2^* + C_3D_3^*$, then (3.8) implies that $A_{11} = 1/2(2s_0 + 1 - 2p)d_7$, $A_{21} = 1/2(1 - 2p)(p - s_0)d_7$, $B_{22} = (1 - q)(q - s_1)d_5$ and $C_{12} = 1/2(2s_1 + 1 - 2q)d_5$. If d_7 is not to be zero then the above equations imply that $p = 1/2$ and $s_0 = 0$. However this would make (3.4) become $1/2 = \frac{4+4s_1^2}{16-4s_1^2}$ which has no solution for $|s_1| < 1$. Likewise if d_5 is not equal to zero then $q = 1$ and $s_1 = -1/2$ which is outside the region of admissible pairs. Therefore $d_5 = d_7 = 0$. But in order for $\langle \psi^1, \psi^2 \rangle = 0$ either d_4 or d_6 must be equal to zero. Hence one of them must have a support of length 1 which is impossible by Lemma 4.2. $\square$

Although it has been shown by Daubechies [5] that with one scaling function compactly supported continuous wavelets cannot be symmetric, this is not the case with two scaling functions. In order to obtain wavelets supported on [0,2] that are symmetric or antisymmetric with respect to one it must be that $s_0 = s_1$. To see this suppose that ψ is such a wavelet with support [0,2]. Since $\psi \in \tilde{V}_1$ it follows using (2.2) and (2.3) that for $x \in [0, \frac{1}{2}]$, $\psi(\frac{x}{2}) = a_0 x + s_0\psi(x)$, and $\psi(2 - \frac{x}{2}) = -a_8 x + s_1\psi(2 - x)$. The symmetry of ψ implies that $\psi(\frac{1}{2^n}) = \pm\psi(2 - \frac{1}{2^n})$ where the plus sign is used if ψ is symmetric with respect to one and the minus sign if it is antisymmetric with respect one. Comparing these equations we see that $s_0 = s_1$ which implies that both s values are equal to $-1/5$. Suppose that one wavelet ψ^s is symmetric while the other ψ^a is antisymmetric with respect to one and set, $\bar{y}_1^0 = [0, a_1^0, a_2^0]$, $\bar{y}_2^0 = [a_2^0, a_3^0, a_4^0]$, $\bar{y}_3^0 = [a_4^0, a_3^0, a_2^0]$, $\bar{y}_4^0 = [a_2^0, a_1^0, 0]$, and $\bar{y}_1^1 = [0, a_1^1, a_2^1]$, $\bar{y}_2^1 = [a_2^1, a_3^1, 0]$, $\bar{y}_3^1 = [0, -a_3^1, -a_2^1]$, $\bar{y}_4^1 = [-a_2^1, -a_1^1, 0]$, then equations similar to (4.2)–(4.5), ((4.6) is automatically satisfied) give $a_1^0 = 1$, $a_2^0 = -30$, $a_3^0 = 111$, $a_4^0 = -100$, and $a_1^1 = 1$, $a_2^1 = -30$, $a_3^1 = 81$. Here $\|\psi^s\|_{L^2} = \sqrt{2}\|\psi^a\|_{L^2}$. If ψ^s and ψ^a



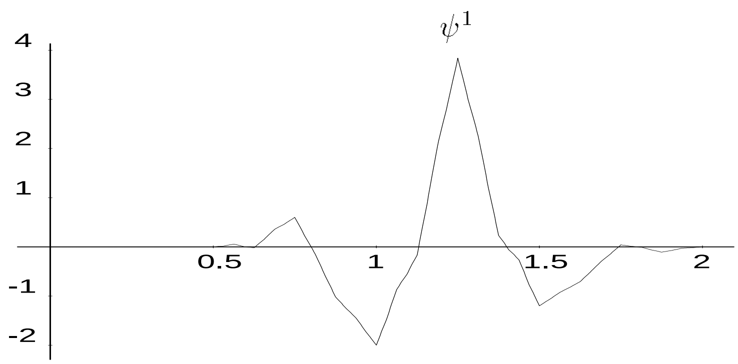


Figure 4

normalize to 1 then the corresponding matrices in (4.10) are

$$D_0 = \begin{bmatrix} -1/20 & -3\sqrt{2}/20 \\ -\sqrt{2}/20 & -3/10 \end{bmatrix}, \qquad D_1 = \begin{bmatrix} 9/20 & -1/\sqrt{2} \\ 9\sqrt{2}/20 & 0 \end{bmatrix},$$
$$D_2 = \begin{bmatrix} 9/20 & -3\sqrt{2}/20 \\ -9\sqrt{2}/20 & 3/10 \end{bmatrix}, \qquad D_3 = \begin{bmatrix} -1/20 & 0 \\ \sqrt{2}/20 & 0 \end{bmatrix}.$$

These wavelets are plotted in Figure 5. Consequently ψ^s and ψ^a individually exhibit linear phase.

A similar computation where both wavelets are assumed to be symmetric with respect to one and supported on [0,2] yields no solution.

Finally we note that the multiresolution analysis arising from AFIF is well suited for compact intervals. In fact, if we let $V_k' = \tilde{V}_k \cap L^2[0,1]$ and $\tilde{\phi}^i_{k,j} = \phi^i_{k,j}|_{[0,1]}$ then $\{V_k'\}$

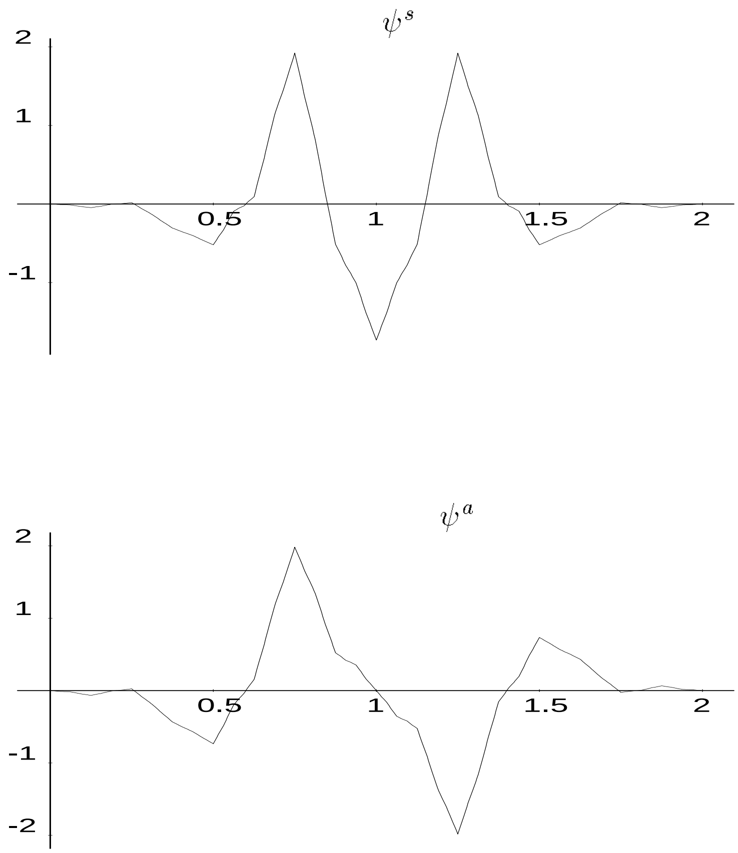


Figure 5

provides a multiresolution analysis for $[0,1]$ and $\tilde{\phi}^i_{k,j}$, $j \in \mathbb{Z}$, $i = 0,1$ is an orthogonal basis for V'_k. To see what to do with the wavelets write $\psi^i_j = \psi^i|_{[j-1,j]}$, and $\phi^i_j = \phi^i|_{[j-1,j]}$, $i = 0,1$, $j = 1,2$. We now rotate the wavelets i.e.,

$$\psi^{+,0} = a\psi^0 + b\psi^1, \quad \psi^{+,1} = -b\psi^0 + a\psi^1 \tag{4.26}$$

with $|a|^2 + |b|^2 = 1$ so that

$$\langle \psi^{+,0}_j, \phi^1_j \rangle = 0, \qquad j = 1,2. \tag{4.27}$$

To see that this can be done note that (4.27) is equivalent to

$$a\langle \psi^0_1, \phi^1_1 \rangle + b\langle \psi^1_1, \phi^1_1 \rangle = 0, \tag{4.28}$$

and

$$a\langle\psi_2^0,\phi_2^1\rangle + b\langle\psi_2^1,\phi_2^1\rangle = 0. \tag{4.29}$$

However the fact that $\langle\psi^0,\phi^1\rangle = \langle\psi_1^0,\phi_1^1\rangle+\langle\psi_2^0,\phi_2^1\rangle = 0$ and $\langle\psi^1,\phi^1\rangle = \langle\psi_1^1,\phi_1^1\rangle+\langle\psi_2^1,\phi_1^1\rangle = 0$ shows us that (4.28) and (4.29) are not independent which allows us to construct the desired rotation. With $\psi^{+,0}$ and $\psi^{+,1}$ constructed as in (4.26) set $\tilde{\psi}_{k,j}^0 = \psi_{k,j}^{0,+}|_{[0,1]}$ and

$$\tilde{\psi}_{k,j}^1 = \begin{cases} 0 & \text{if } \psi_{k,j}^{+,1} \cap [0,1]^c \neq \emptyset \\ \psi_{k,j}^{+,1} & \text{otherwise} \end{cases} \tag{4.30}$$

then the non-zero components of $\{\tilde{\Psi}_{k,j}\}_{j\in\mathbb{Z}}$ form an orthogonal basis for W_k', $k \geq 0$ where $V_{k+1}' = V_k' \oplus W_k'$. This leads to

Theorem 4.4. *$\{\tilde{\phi}_{k,j}^i = \phi_{k,j}^i|_{[0,1]}, k \geq 0, i = 0,1, -i \leq j \leq 2^k - 1\}$ is an orthogonal basis for $V_k' = V_k \cap L^2[0,1]$ while $\{\tilde{\psi}_{k,j}^i, k \geq 0, i = 0,1, i-1 \leq j \leq 2^k-(i+1)\}$ forms an orthogonal basis for W_k'. Furthermore $\bigoplus_{k\geq 0} W_k' = L^2[0,1]$.*

For the case where $s_0 = s_1 = -1/5$ it is easy to see that $\tilde{\psi}_{k,j}^0$ are just the symmetric wavelets restricted to $[0,1]$ i.e., $\tilde{\psi}_{k,j}^0 = \psi_{k,j}^s|_{[0,1]}$ while $\tilde{\psi}_{k,j}^1 = \psi_{k,j}^a$ if the support of $\psi_{k,j}^a \subset [0,1]$ and 0 otherwise.

V. Scaling by Other Integers

Many of the results of the previous sections can be used to produce scaling functions and wavelets satisfying dilation equations of the form

$$\Phi(x) = \sum C_i \Phi(Nx - i),$$

and

$$\Psi(x) = \sum D_i \Phi(Nx - i),$$

with $N > 2$. Note that in this case the matrices D_i will not in general be square matrices. We begin by considering the $N+1$ dimensional basis $\{f_{\bar{y}_i}\}_{i=0}^N$ spanning S_0 where $\bar{y}_i = e_i$, $0 < i \leq N$, $\{e_i\}_{i=0}^N$ being the standard basis in R^{N+1} and last vector $\bar{y}_0$ is given by $\bar{y}_0 = [1, q_1, q_2, \ldots, q_{N-1}, 0]$. What is needed is to adjust q_i, $1 \leq i \leq N-1$ and s_j, $0 \leq j \leq N$ so that $f_{\bar{y}_0}$ is nonzero and orthogonal to $f_{\bar{y}_i}$, $1 \leq i \leq N$. Once this has been accomplished, orthogonal scaling functions ϕ^i, $0 \le i \le N-1$ can be obtained by applying the Gram-Schmidt procedure to the set $\{f_{\bar{y}_i}\}_{i=1}^{N}$. The functions ϕ^i, $0 \le i \le N-2$ obtained from the functions $\{f_{\bar{y}_i}\}_{i=1}^{N-1}$ will be continuous and supported on $[0,1]$ since each $f_{\bar{y}_i}$, $1 \le i \le N-1$ vanishes at zero and one. The last function ϕ^{N-1} can be obtained by subtracting from $f_{\bar{y}_N}$ its projection onto the subspace spanned by $\{\phi^i\}_{i=0}^{N-2}$ then piecing it together continuously with $f_{\bar{y}_0}$ as was done in the case $N=2$ in Section 3. That it is sufficient to consider only a basis for S_0 of this type follows from Theorem 5.3.

Since there are N orthogonality relations and $2N-1$ unknowns (q_i, $1 \le i \le N-1$, and s_j, $0 \le j \le N-1$), it may be possible to impose other desirable conditions besides orthogonality and still obtain the required basis $\{\phi^i\}_{i=0}^{N-1}$. If $\Phi^*(x) = (\phi^1, \phi^2, \ldots, \phi^{N-1})$ then Φ will satisfy $\Phi(x) = \sum_{i=0}^{2N-1} C_i \Phi(Nx - i)$. $\Psi(x)$ may now be obtained from $\Phi(x)$ using Theorem (4.1) or the orthogonality equations (2.9) and solving as in Section 4.

In order to compute the orthogonality relations $\langle f_{\bar{y}_i}, f_{\bar{y}_0}\rangle = 0$, $1 \le i \le N$, $\lambda_j^i(x) = a_j^i x + b_j^i$, $0 \le i \le N$, $0 \le j \le N-1$ need to be computed. From (2.1), (2.2), (2.3) and (2.5) we find that $a_j^i = \delta_{i-1,j} - \delta_{i,j} - s_j \delta_{N,i}$ and $b_j^i = \delta_{i,j}$ for $1 \le i \le N$ and $0 \le j \le N-1$ with $q_0 = 1$ and $q_N = 0$. Also $a_j^0 = q_{j+1} - q_j + s_j$, $b_j^0 = q_j - s_j$ for $0 \le j \le N-1$. If $S_1 = \sum_{i=0}^{N-1} s_i$ and $S_2 = \sum_{i=0}^{N-1} i s_i$ it follows from (2.7) and (2.8) that $m_0^i = \frac{1}{N-S_1}$, $m_1^i = \frac{(N-S_1)i + S_2}{(N-S_1)(N^2-S_1)}$ for $1 \le i \le N-1$, $m_0^N = \frac{1-S_1}{2(N-S_1)}$,

$$m_1^N = \frac{N(3N-1) + (1-5N)S_1 + 3(1-N)S_2 + 2S_1^2}{6(N-S_1)(N^2-S_1)},$$

$$m_0^0 = \frac{1 - S_1 + 2\sum_{i=1}^{N-1} q_i}{2(N-S_1)},$$

and

$$m_1^0 = \frac{S_1(S_1 - (N+1)) - 3(N-1)S_2 + N + 6\sum_{i=1}^{N-1} q_i((N-S_1)i + S_2)}{6(N-S_1)(N^2-S_1)}.$$

With the above moments (2.9) yields

$$
\begin{aligned}
I_{n,0} &= \langle f_{\bar{y}_n}, f_{\bar{y}_0} \rangle \\
&= \Bigg((s_{n-1} - s_n) m_1^0 + (m_0^n - m_1^n) \sum_{i=0}^{N-1} s_i (q_i - s_i) \\
&\qquad + m_1^n \sum_{i=1}^{N-1} s_{i-1} q_i + s_n m_0^0 - \frac{s_{n-1} + 2 s_n}{6} \\
&\qquad + \frac{1}{6} (q_{n-1} + 4 q_n + q_{n+1}) \Bigg) \Bigg/ \left(N - \sum_{i=0}^{N-1} s_i^2 \right), \qquad 1 \leq n \leq N-1,
\end{aligned} \tag{5.1}
$$

and

$$
\begin{aligned}
I_{N,0} = \Bigg(& m_1^0 \left(s_{N-1} - \sum_{i=0}^{N-1} s_i^2 \right) + (m_0^N - m_1^N - 1/6) \sum_{i=0}^{N-1} s_i (q_i - s_i) \\
& + \sum_{i=1}^{N-1} s_{i-1} q_i (m_1^N - 1/3) + \frac{q_{N-1} - s_{N-1}}{6} \Bigg) \Bigg/ \left(N - \sum_{i=0}^{N-1} s_i^2 \right).
\end{aligned} \tag{5.2}
$$

For $N = 2$ (5.1) and (5.2) yield (3.2) and (3.3) with $p_1 = 0$. For $N = 3$ (5.1) and (5.2) become more complicated and we shall restrict ourselves to considering s values that give symmetric or antisymmetric wavelets.

Just as when $N = 2$, we require that the s-values be arranged symmetrically, so s_0 must be the same as s_2. In this case a one parameter family of continuous, compactly supported, orthogonal scaling functions each with linear phase will be produced.

We proceed as before, examining what conditions S_0 must satisfy for compactly supported scaling functions to exist. Let Υ_0 be the space of vectors vanishing on both sides, and Υ_1 be those vanishing on the left. Then Υ_0 is a 2-dimensional space, and has a basis of the form $\bar{x}_1 = [0, 1, 1, 0]$ and $\bar{x}_2 = [0, 1, -1, 0]$. Let $\bar{x}_3 = [0, p_1, p_2, 1]$ be a vector orthogonal to Υ_0 in Υ_1, so that $\{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$ forms an orthogonal basis for Υ_1.

Now, if we can find $\bar{x}_4 = [1, q_1, q_2, 0]$ orthogonal to Υ_1, then we easily generate the scaling functions

$$
\phi^0 = \begin{cases} f_{\bar{x}_1} & \text{on } [0,1] \\ 0 & \text{elsewhere} \end{cases} \qquad \phi^1 = \begin{cases} f_{\bar{x}_2} & \text{on } [0,1] \\ 0 & \text{elsewhere} \end{cases} \qquad \text{and} \quad \phi^2 = \begin{cases} f_{\bar{x}_3} & \text{on } [0,1] \\ f_{\bar{x}_4}(\cdot - 1) & \text{on } (1,2] \\ 0 & \text{elsewhere.} \end{cases}
$$

If there exists no such $\bar{x}_4$, then there are no compactly supported scaling functions as we can see from the following general result

Theorem 5.1. *Let V be an N-dimensional subspace of $C^0[0,1]$ such that there does not exist an orthonormal basis with $N-1$ of the basis functions vanishing at zero and the remaining function vanishing at one, or vice versa. Then there does not exist compactly supported $C^0(\mathbb{R})$ functions $\{\phi^j\}_{j=1}^{N-1}$ composed of linear combinations of basis elements of V and their integer translates constructed so that $\langle \phi^j(x), \phi^i(x-l)\rangle = \delta_{j,i}\delta_{0,l}$.*

Proof. The proof is by induction on N. The case $N=3$ was established in Lemma 3.5. We only consider the step from $N=3$ to $N=4$ as the general argument is similar. For ease of notation set $x=\phi^1, y=\phi^2$, and $z=\phi^3$ and suppose that each is supported on a subset of $[0,M]$. Suppose V satisfies the hypotheses above then there are three cases we need to consider:

Case 1: One of the functions (say, x) is supported on $[0,1]$, or x, y, and z can be transformed by a finite number of rotations and shifts so that this is the case. First, perform this transformation, if necessary. We see that the components of y and z are restricted to a 3-dimensional space, which has no orthonormal basis of vectors with two functions vanishing on the left and one vanishing on the right or vice versa. Thus by Lemmas 3.4 and 3.5, the scaling functions y and z cannot exist.

Case 2: All three functions have support $[0,2]$ or longer, and the leftmost components, $\{x_1, y_1, z_1\}$, have rank 1. In this case, we may begin to apply the rotate and shift strategy of Lemma 3.5 on y and z. If for some finite j, $y_1^{(j)}$ and $z_1^{(j)}$ are not linearly dependent, then we go to Case 3, below. Otherwise, the proof of Lemma 3.5 applies, and we are done.

Case 3: All three functions have support $[0,2]$ or longer, and the leftmost components, $\{x_1, y_1, z_1\}$, have rank 2. Note that $\{x_1, y_1, z_1\}$ cannot have rank 3 because then the rightmost nonzero component of any of them would be orthogonal to all functions vanishing on the left, which we assumed was impossible. Here, we apply an argument similar to the one found in Lemma 3.5 using two rotation maps instead of one. Many of the details in this proof are analogous to those in the proof of Lemma 3.5, so they are omitted here.

Two among x_1, y_1, and z_1 must be linearly independent, so assume y_1 and z_1 are they. We can rotate y and z so that y_1 and z_1 are orthogonal, and for an appropriate basis, we have

$$x = ((x_{1,1}, x_{1,2}, 0, 0), x_2, x_3, \ldots, x_M)$$
$$y = ((y_{1,1}, 0, 0, 0), y_2, y_3, \ldots, y_M)$$
$$z = ((0, z_{1,2}, 0, 0), z_2, z_3, \ldots, z_M)$$

where $y_{1,1} > 0$ and $z_{1,2} > 0$. As before, we have a rotation r_1 which transforms x and y into

$$x' = ((0, x'_{1,2}, 0, 0), x'_2, \ldots, x'_M)$$
$$y' = ((y'_{1,1}, y'_{1,2}, 0, 0), y'_2, \ldots, y'_M)$$

leaving z unchanged and r_2 which transforms x' and z into

$$x'' = (0, x''_2, \ldots, x''_M)$$
$$z' = ((0, z'_2, 0, 0), z'_2, \ldots, z'_M)$$

leaving y' unchanged. We also have the shift map s, which takes x'' to $(x''_2, x''_3, \ldots, x''_M, 0) = x'''$ leaving y' and z' unchanged. The resulting vectors x''', y', and z' are themselves orthogonal scaling functions, and thus the leftmost components cannot have rank 3. Clearly, y'_1 and z'_1 are linearly independent, so their rank is 2. Now, we iterate the map $s \circ r_2 \circ r_1$ to obtain the sequence $(x^{(j)}, y^{(j)}, z^{(j)}) = (s \circ r_2 \circ r_1)^j(x, y, z)$, which must have some limit point, (X, Y, Z). Analogously to Lemma 3.5, both $\{y^{(j)}_{1,1}\}$ and $\{z^{(j)}_{1,2}\}$ are monotone increasing from which it follows that $X_{k,1} = X_{k,2} = 0$ for $k = 1, 2, \ldots, M$ and hence that $X = 0$, which we know is not possible.

To proceed to higher N we need only consider the analog of Case 3 above since the other cases are eliminated by the induction hypothesis. The analog of Case 3 is when all N functions have support $[0, 2]$ or longer and the dimension of the space spanned by the leftmost components of these functions is $N - 2$. In this case $N - 2$ rotations are needed to reduce by iteration one of the functions to zero thereby forcing a contradiction. $\square$

Next, we investigate what conditions must be imposed on s_0 and s_1 in order that such a basis should exist. The vectors $[0, 1, 0, 0]$, $[0, 0, 1, 0]$, and $[0, 0, 0, 1]$ form a basis (not orthonormal) for Υ_1, so it suffices to check for orthogonality between $\bar{x}_3$ and each of these vectors. From (5.1) and (5.2) we find (where $\bar{y}_i, i = 0, 1, 2, 3$ are used), after some manipulation,

$$
\begin{aligned}
I_{1,0} = {} & 8s_0s_1q_2 - 108q_1 - 27q_2 + 12s_0 + 42s_1 + 60s_0q_1 - 48s_0q_2 - 8s_0s_1 \\
& - 24s_1q_1 - 24s_1q_2 + 76s_0^2 + 31s_1^2 + 8s_1s_0q_1 - 24s_0^3 - 6s_1^3 - 27 \\
& + 20s_0^2q_2 - 12s_0^2s_1 - 12s_0s_1^2 - s_1^2q_2 + 8s_1^2q_1 - 16q_1s_0^2 = 0, \\
I_{2,0} = {} & 8s_0s_1q_2 - 27q_1 - 108q_2 + 12s_0 + 24s_1 - 48s_0q_1 + 60s_0q_2 - 8s_0s_1 \\
& - 24s_1q_1 - 24s_1q_2 + 28s_0^2 + 16s_1^2 + 8s_1s_0q_1 - 16s_0^2q_2 \\
& + 8s_1^2q_2 - s_1^2q_1 + 20q_1s_0^2 = 0, \\
I_{3,0} = {} & -8s_0s_1q_2 - 27q_2 + 48s_0 + 12s_0q_1 + 12s_0q_2 - 4s_0s_1 + 24s_1q_1 \\
& + 42s_1q_2 - 50s_0^2 - 21s_1^2 - 8s_1s_0q_1 - 12s_0^2s_1q_2 - 32s_0^3 - 8s_1^3 \\
& + 76s_0^2q_2 - 16s_0^2s_1 - 16s_0s_1^2 + 31s_1^2q_2 + 16s_1^2q_1 + 28q_1s_0^2 \\
& - 24s_0^3q_2 + 8s_0^3s_1 + 6s_0^2s_1^2 + 4s_1^3s_0 - 6s_1^3q_2 - 12s_1^2s_0q_2 \\
& + 8s_0^4 + s_1^4 = 0.
\end{aligned}
$$

Solving the first two for q_1 and q_2 gives

$$
q_1 = \frac{32s_0^3 - 76s_0^2 - 32s_0^2s_1 + 24s_0s_1 + 16s_0s_1^2 - 24s_1 + 36 - 56s_1^2 - 16s_1^3}{(6s_0 + 21s_1 + 45)(2s_0 - s_1 - 3)},
$$

$$
q_2 = \frac{40s_0^3 - 20s_0^2 - 4s_0^2s_1 + 12s_0s_1 + 20s_0s_1^2 - 36s_0 - 30s_1 - 19s_1^2 - 9 - 2s_1^3}{(6s_0 + 21s_1 + 45)(2s_0 - s_1 - 3)}.
$$

If we substitute these expressions into the third equation and simplify, we get the desired relationship,

$$
\begin{aligned}
0 = {} & 48s_0^4 - 256s_0^3 + 16s_0^2s_1^2 + 192s_0^2s_1 + 400s_0^2 + 16s_0s_1^3 - 96s_0s_1^2 \\
& - 272s_0s_1 - 192s_0 + s_1^4 + 16s_1^3 + 70s_1^2 + 48s_1 + 9.
\end{aligned}
$$

Let the polynomial on the right be denoted by $\hat{p}(s_0, s_1)$. The zero set of $\hat{p}$ is shown in Figure 6. From the above, it is now clear that

Theorem 5.2. *Continuous, compactly supported, orthogonal scaling functions $\{\phi^j\}_{j=0}^2$ can be constructed so that this set and its integer translates span $\tilde{V}_0$ $(N = 3, s_2 = s_0)$ if and only if $|s_0| < 1$, $|s_1| < 1$, and $\hat{p}(s_0, s_1) = 0$.*

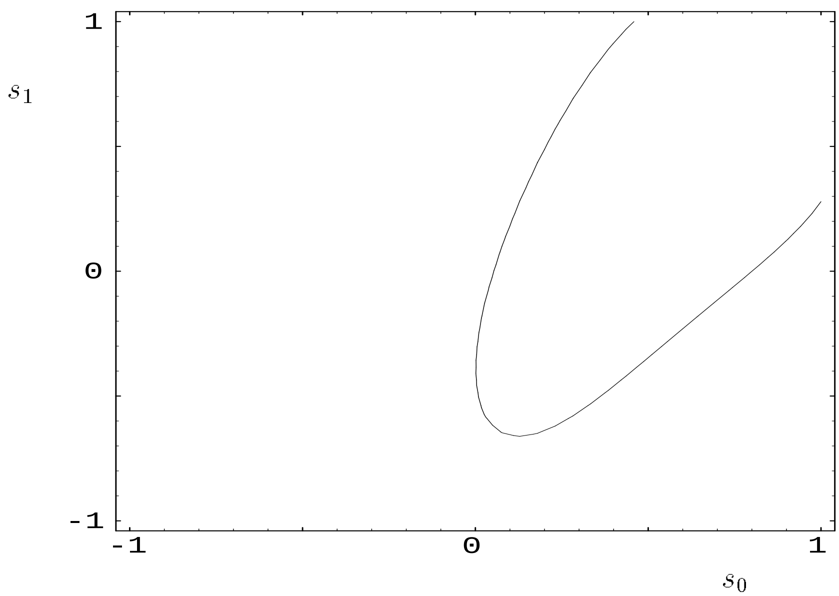


Figure 6

As an example, we construct the scaling functions and wavelets generated from $s_0 = s_2 = \frac{3}{7}$, $s_1 = -\frac{3}{7}$. It is easy to check that these values do satisfy the conditions of Theorem 5.3, so the construction above gives the scaling functions shown in Figure 7. We then generate the wavelets, which are shown in Figure 8. The interpolation values for these functions are given in Tables 2 and 3. Table 4 gives the matrices in the dilation equation for the scaling functions obtained from $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4$, ($s_0, s_1, s_2$ arbitrary) given above.

For general N we consider two special cases: Case 1, $s_i = 0$, $0 \le i \le N-2$, $s_{N-1} = s$ and Case 2, $s_i = s$, $0 \le i \le N-1$.

For Case 1, $S_1 = s$ and $S_2 = (N-1)s$, therefore $m_0^0 = \frac{(1-s+2\sum_{i=1}^{N-1} q_i)}{2(N-s)}$, and

$$m_1^0 = \frac{(s^2 - s(3N^2 - 5N + 4)) + N + 6\sum_{i=1}^{N-1} q_i((N-s)i + (N-1)s)}{6(N^2 - s)(N-s)}.$$

and (5.1) yields

$$I_{n,0} = \frac{(m_0^n - m_1^n)s(q_{N-1} - s)) + \frac{1}{6}(q_{n-1} + 4q_n + q_{n+1})}{N - s^2}. \qquad 1 \le n \le N-2 \tag{5.3}$$

With the benefit of hindsight we set $q_{N-1} = s$ and solve,

$$I_{n,0} = q_{n-1} + 4q_n + q_{n+1}. \qquad 1 \le n \le N-2 \tag{5.4}$$

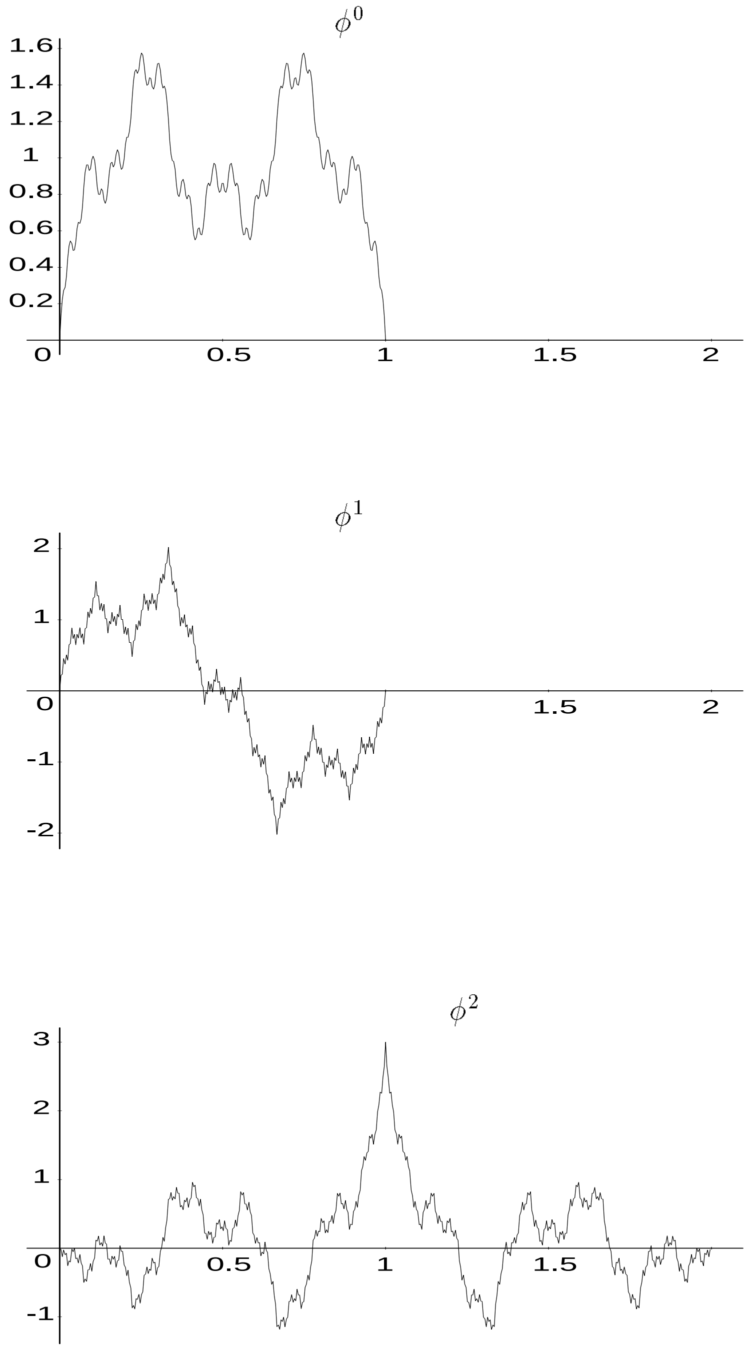

ϕ^0
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.5
1
1.5
2
ϕ^1
2
1
0
1.5
2
-1
-2
ϕ^2
3
2
1
0
0.5
1
1.5
-1

Figure 7

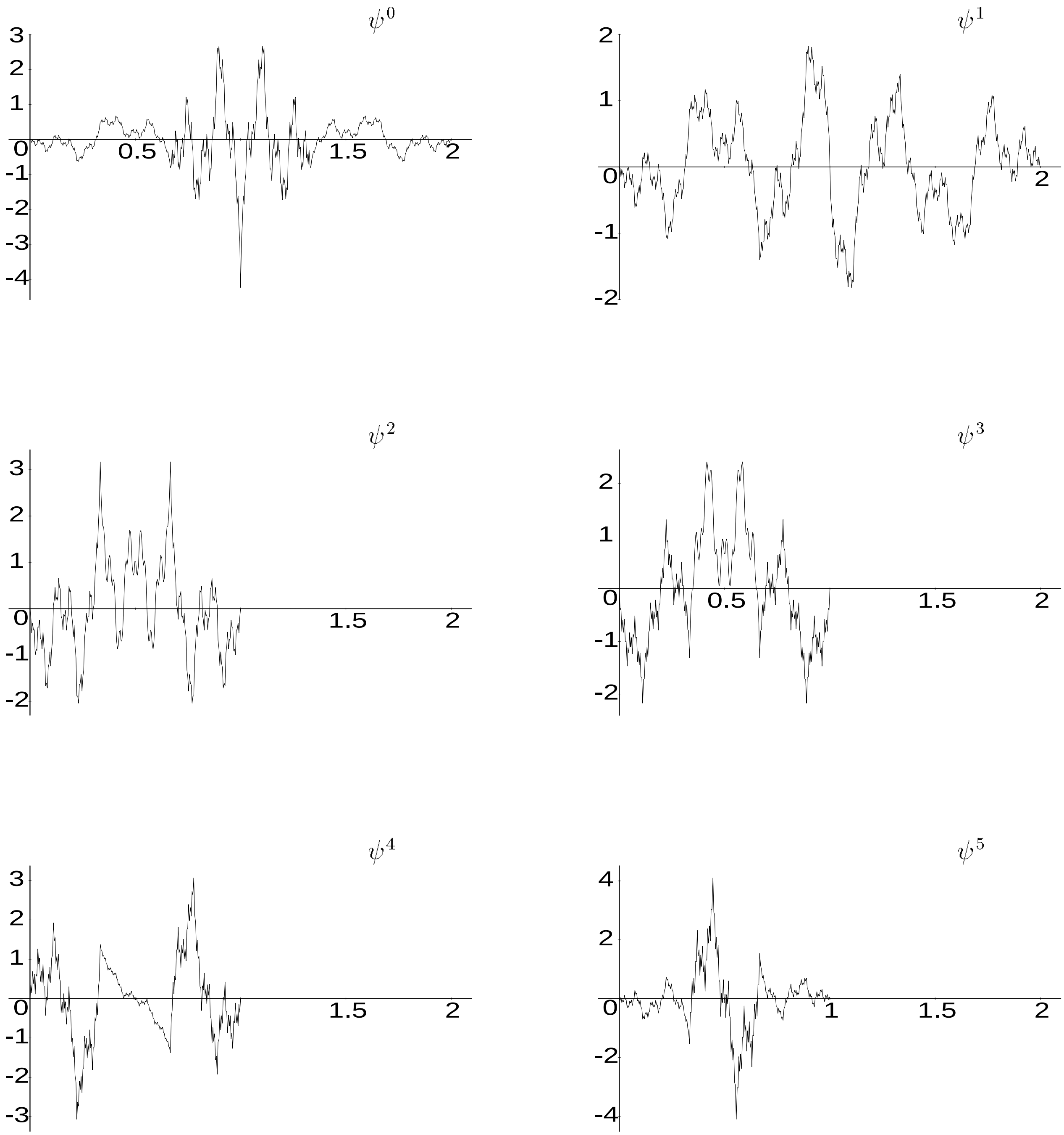

ψ0
3
2
1
0
-1
-2
-3
-4
0.5
1.5
2
ψ1
2
1
0
-1
-2
2
ψ2
3
2
1
0
-1
-2
1.5
2
ψ3
2
1
0
-1
-2
0.5
1.5
2
ψ4
3
2
1
0
-1
-2
-3
1.5
2
ψ5
4
2
0
-2
-4
1
1.5
2

Figure 8

with boundary conditions $q_0 = 1$ and $q_{n-1} = s$. The solution is

$$q_n = \frac{U_{n-1} s}{U_{N-2}} + \frac{U_{N-n-2}}{U_{N-2}},$$

where U_i is the Chebyshev polynomial of the second kind evaluated at $x = -2$ i.e., $U_i = \frac{\lambda^{i+1} - \lambda^{-(i+1)}}{\lambda - \lambda^{-1}}$ with $\lambda = -2 + \sqrt{3}$. Using (5.4) we also find that

$$\sum_{n=1}^{N-1} q_n = \frac{1}{6}(5q_{N-1} + q_{N-2} + q_1 - 1), \tag{5.5}$$

and

$$\sum_{n=1}^{N-1} nq_n = \frac{((5N-4)q_{N-1} + (N-1)q_{N-2} - 1)}{6}. \tag{5.6}$$

To obtain $I_{N-1,0} = 0$ with $q_{N-1} = s$ we find from (5.1) that

$$s(m_1^0 + m_0^0 - \frac{1}{3}) + \frac{1}{6}(q_{N-2} + s) = 0. \tag{5.7}$$

Likewise from (5.2) we find that $I_{N,0} = 0$ only if $s(s-1)m_1^0 = 0$. Since $s \neq 0$ or 1 we find

$$m_1^0 = 0. \tag{5.8}$$

Substitute (5.5) and (5.8) into (5.7), and (5.5) and (5.6) into (5.8), to find that (5.7) and (5.8) are both equal to zero if and only if s satisfies the equation

$$s^2 + (N+1)(U_{N-3} + 2U_{N-2})s + N = 0, \qquad N \geq 2. \tag{5.9}$$

This leads to,

Theorem 5.3. *Suppose $N \geq 2, s_i = 0, 0 \leq i \leq N-2$ and if $s_{N-1} = s = \frac{-b+\sqrt{b^2-N}}{2}$ with $b = (N+1)(U_{N-3} + 2U_{N-2})$ with $U_i = \frac{\lambda^{i+1} - \lambda^{-(i+1)}}{\lambda - \lambda^{-1}}$ with $\lambda = -2 + \sqrt{3}$. Then there exist continuous, compactly supported, orthogonal scaling functions $\{\phi^i\}_{i=0}^{N-1}$ supported on $[0, 2]$ such that this set and its integer translates span $\tilde{V}_0$.*

A set of mother wavelets $\{\psi^j\}_{j=1}^{N(N-1)}$ may be constructed using Theorem (4.1) or the orthogonality equations (2.9) .

For Case 2, $S_1 = Ns$ and $S_2 = \frac{N(N-1)s}{2}$ therefore $m_0^n = \frac{1}{N(1-s)}$, $m_1^n = \frac{s(N-2n-1)+2n}{2N(1-s)(N-s)}$, $1 \le n \le N-1$, $m^N = \frac{1-Ns}{2N(1-s)}$, $m_1^N = \frac{4Ns^2-s(3N+4N+1)+6N-2}{12N(1-s)(N-s)}$, $m_0^0 = \frac{1-Ns+2a}{2N(1-s)}$, and

$$m_1^0 = \frac{2Ns^2 - s(3N^2-4N+5)+2+12(1-s)a+6(N-1)sb}{12N(1-s)(N-s)},$$

where $a = \sum_{i=1}^{N-1} q_i$ and $b = \sum_{i=1}^{N-1} iq_i$. The orthogonality conditions become

$$\begin{aligned} I_{n,0} = 0 = {} & s[s^2(N^2 - N(2n-1)) - 2s(N^2 - n(N+1)+1) \\ & + 3N - N^2 - 2n] + 4(N-s)sa \\ & + \frac{N(1-s)(N-s)}{3}(q_{n-1}+4q_n+q_{n+1}), \qquad 1 \le n \le N-1, \end{aligned} \tag{5.9}$$

and

$$\begin{aligned} I_{N,0} = 0 = {} & m_1^0 s(1-Ns) + (m_0^N - m_1^N - 1/6)s(a - Ns + 1) \\ & + sa(m_1^N - 1/3) + \frac{q_{N-1}-s}{6}, \end{aligned} \tag{5.10}$$

with $n = N-1$ in (5.9) and we find

$$a = \frac{s^3(N^2-3N)+4s^2+s(N^2-N-2) - \frac{N}{3}(1-s)(N-s)(q_{N-2}+4q_{N-1})}{4s(N-s)}. \tag{5.11}$$

Therefore (5.9) becomes for $1 \le n \le N-2$

$$0 = q_{n-1} + 4q_n + q_{n+1} - d_n \tag{5.12}$$

where $d_n = (q_{N-2}+4q_{N-1}) + \frac{6}{N(N-s)}s(sN-1)(N-n-1)$. With $\bar{q}^* = (q_1, q_2, \ldots, q_{N-2})$, H the $N-2 \times N-2$ tridiagonal matrix with 4 on the diagonal and 1 on the off diagonal, and $\bar{c}^* = (d_1 - 1, d_2, \ldots, d_{N-3}, d_{N-2} - q_{N-1})$ we find that

$$\bar{q} = H^{-1}\bar{c}. \tag{5.13}.$$

It follows from $HH^{-1} = I$ that

$$H_{n,m}^{-1} = \begin{cases} -\frac{U_{n-1}U_{N-2-m}}{U_{N-3}}, 0 \le n \le m \le N-1 \quad N \ge 3, \\ H_{n,m}^{-1} = H_{m,n}^{-1}, 0 \le m \le n \le N-1. \end{cases} \tag{5.14}$$

Here U_i is as in Case 1 above. The above equation gives $q_i, 1 \le i \le N-2$ in terms of q_{N-2} and q_{N-1}. Since $H_{N-2,i}^{-1} = -\frac{U_{i-1}}{U_{N-3}}$ we find that $\sum_{i=1}^{N-2} H_{N-2,i}^{-1} = \frac{1}{6}\frac{U_{N-2}-U_{N-3}-1}{U_{N-3}}$. Solving (5.13) for q_{N-2} then substituting in the above calculations yields

$$q_{N-2} = \frac{(4-2U_{N-3}-6U_{N-2})q_{N-1} + 6 - \frac{36s(sN-1)}{N(N-s)}\sum_{i=1}^{N-2} U_{i-1}(N-n-1)}{5U_{N-3}+U_{N-2}-1}.$$

To solve for q_{N-1} sum Equation (5.9) from $n=0$ to $N-2$ to find

$$a = \frac{s^2(3\,N^2 - 3\,N) + s(3\,N^2 - 8\,N) + 6}{N(6\,sN - 12\,s + 6\,N)} - \frac{(q_{N-1} + q_1)\,(s-1)\,N}{6\,sN - 12\,s + 6\,N}.$$

Eliminate q_1 from this equation using (5.13) then comparing it with (5.11) gives a solution for q_{N-1}.Using a symbolic manipulation language such as Maple to solve these equations and using the recurrence formula for Chebyshev polynomials of the second kind we find

$$\begin{aligned}
q_{N-2} = &-\frac{s^3 N((21\,N^2 - 93\,N + 104\,)U_{N-3} + (6\,N^2 - 24\,N + 28\,)U_{N-4} - 6\,N^2 + 12\,N + 4)}{d} \\
&+ \frac{2\,s^2((45\,N^2 - 45\,N + 52\,)U_{N-3} + (12\,N^2 12\,N + 14\,)U_{N-4} + 6\,N^3 + 2)}{d} \\
&+ \frac{3\,sN((7\,N^2 - N - 30\,)U_{N-3} + (2\,N^2 - 8\,)U_{N-4} - 2\,N^2 + 4\,N) - 12\,N^3}{d}
\end{aligned}$$

and

$$\begin{aligned}
q_{N-1} = &\frac{-s^3 N(4 + 3\,N^2 - 9\,N) - 2\,s^2(3\,N^3 - 2) + 3\,sN^2(N-3) + 6\,N^3}{d} \\
&+ \frac{U_{N-4}(s^3 N(213\,N^2 - 369\,N + 388) - s^2(336\,N^2 - 336\,N + 388) - sN(213\,N^2 - 33\,N - 336))}{d} \\
&- \frac{U_{N-5}(s^2(90\,N^2 - 90\,N + 104) - s^3 N(57\,N^2 - 99\,N + 104) + sN(57\,N^2 - 9\,N - 90))}{d}
\end{aligned}$$

where $d = 2\,N(s-N)(s(2 + U_{N-4}(168\,N - 194) + U_{N-5}(45\,N - 52)) + N(168\,U_{N-4} + 45\,U_{N-5}))$. Note that d is non-zero for $-1 \le s \le 1$. What remains is to solve (5.10). If we multiply (5.9) by $n-1$, sum for $n = 1$ to $N-1$ then solve for $\sum_{n=0}^{N-1} nq_n = b$ we find after using (5.11),

$$b = \frac{s^2(N^3 - 3N^2 + 2N) - s(N^2 - 3N + 1) - N}{6\,N - 6\,s} + \frac{(2\,N-1)\;N q_{N-1}}{6}.$$

Substituting the above equations in (5.10) and simplifying using the recurrence formula for the Chebychev polynomials yields,

$$\begin{aligned}
0 = &\frac{N(s^3(U_{N-4}(35\,N^2 - 12\,N^3 + 35\,N^2 - 36\,N - 7) + U_{N-3}(-45\,N^3 + 130\,N^2 - 135\,N - 26) - N^2 - 1)}{d_1} \\
&+ \frac{s^2(U_{N-4}(-12\,N^3 + 45\,N^2 + 2\,N + 21) + U_{N-3}(7\,N - 45\,N^3 + 168\,N^2 + 7\,N + 78) - 3\,N^2 + 2\,N + 3)}{d_1} \\
&+ \frac{-6\,sN((3\,U_{N-4} + 11\,U_{N-3})(N+1) + 1) + 6\,N^2)(s-1)(s+1)}{d_1}
\end{aligned}$$

which is equal to zero for $|s| < 1$ when

$$\begin{aligned} p_N(s) &= s^3(U_{N-4}(35\,N^2 - 12\,N^3 - 36\,N - 7) + U_{N-3}(130\,N^2 - 45\,N^3 - 135\,N - 26) - N^2 - 1) \\ &+ s^2(U_{N-4}(21 - 12\,N^3 + 2\,N + 45\,N^2) + U_{N-3}(7\,N + 168\,N^2 - 45\,N^3 + 78) + 3 + 2\,N - 3\,N^2) \\ &- 6\,sN(3\,U_{N-4}(N+1) + 11\,U_{N-3}(N+1) + 1) + 6\,N^2 = 0. \end{aligned}$$

Here

$$d_1 = s(24 + (2016\,N - 2328)U_{N-4} + (540\,N - 624)U_{N-5}) + 2016\,U_{N-4}N + 540\,U_{N-5}N)(s-N)^2.$$

Note that d_1 is non-zero for $-1 \leq s \leq 1$. To see that the above cubic has a zero for $|s| < 1$, we evaluate the polynomial p_N at $s = \pm 1$. Thus

$$p_N(1) = -2\,(N-1)^2\,((45\,N - 26)\,U_{N-3} + (12\,N - 7)\,U_{N-4} - 1)$$

and

$$p_N(-1) = 4\,(N+1)^2\,(1 + 7\,U_{N-4} + 26\,U_{N-3})$$

Since $|U_{N-3}| > |U_{N-4}| \geq 0$ and $|U_{N-3}| \geq 1$ for $N \geq 3$ and since sign $U_{N-3} = (-1)^{N+1}$ we see that sign $p_N(1) = (-1)^N$ while sign $p_N(-1) = (-1)^{N+1}$ for $N \geq 3$.

It now follows from $p_N(0) > 0$ that there is a real root s_N of p_N with $|s_N| < 1$ and sign $s_N = (-1)^{N+1}$, $N \geq 3$.

Thus we have shown

Theorem 5.4. *For $N \geq 3$ and $s_i = s, 0 \leq i \leq N-1$ with s a real root of $p_N, |s| < 1$ there exist continuous, compactly supported, orthogonal scaling functions $\{\phi^i\}_{i=0}^{N-1}$ supported on $[0,2]$ such that this set and its integer translates span $\tilde{V}_0$.*

We now give some examples from the above theorem (here we have factored a constant out of $p_N(s)$). For $N = 3$, $p_3(s) = 9s^3 - 7s^2 + 15s - 1$, $q_1 = -\frac{4}{3}\frac{9s^2+2s-3}{(3s+5)(s-3)}$ and $q_2 = \frac{1}{3}\frac{18s^3-9s^2-22s-3}{(3s+5)(s-3)}$. For $N = 4$, $p_4(s) = 53s^4 + 3s^2 + 51s + 1$, $q_3 = \frac{1}{4}\frac{48s^3-25s^2-61s+2}{(5s+7)(s-4)}$, $q_2 = \frac{1}{2}\frac{(2s+1)(3s+1)}{5s+7}$ and $q_1 = -\frac{3}{4}\frac{4s^3+33s^2+9s-10}{(5s+7)(s-4)}$. Finally for $N = 5$ we have $p_5(s) = 7153s^3 + 3061s^2 + 4595s - 25$.

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Table I

s_0	s_1	p_1	q_1
−.45	.24913840973997585189	−.38009192840012554360	−.12894659319741025282
−.45	.50007955918728063419	−.35582657114704781388	−.023947660348619628508
−.4	.090530851689329739269	−.36870444702962973875	−.19115180858473286044
−.4	.64049995285638175410	−.31094033110232474973	.040636075681553380445
−.35	−.0072635265148628714356	−.35283756141879955795	−.22845520471397257834
−.35	.72178391100872377451	−.27028183540476446288	.081026963799106383553
−.3	−.083203336756332486470	−.33573733406363060725	−.25696121868399870140
−.3	.78288589774512971988	−.23067148400259886260	.11334323161084481474
−.25	−.14618214061458410571	−.31804590990375360555	−.28030023201699694275
−.25	.83269626630597925447	−.19153057613718116605	.14123278956066272834
−.2	−.2	−.30000000000000000000	−.30000000000000000000
−.2	.87501353979083931481	−.15268371309061925454	.16624529998818764227
−.15	−.24668956302990954576	−.28170568236593494170	−.31686046415475332193
−.15	.91187113158837005532	−.11408486745843612530	.18920768747150983345
−.1	−.28747604282460888331	−.26321338049774688720	−.33134872545322789185
−.1	.94449789445249286835	−.075743258091117265968	.21061989468097328770
−.05	−.32315154257329379535	−.24454544031947412926	−.34375214058248711088
−.05	.97369298030356868681	−.037696611328522325658	.23080887479344090984
.0	−.35424868893540940950	−.22570811482256823492	−.35424868893540940950
.05	−.38113082998341069519	−.20669731648304742064	−.36294295781760398380
.1	−.40404262161178188244	.18750151351161996512	−.36988575837081197105
.15	−.42313982625655554203	−.16810316494975289571	−.37508507407448600005
.2	−.43850726493738378110	−.14847932732430920238	−.37851197269001982088
.25	−.45016949520593589167	−.12860173660964793697	−.38010331409967757543
.3	−.45809667756743604588	−.10843651287567490963	−.37976221812377236383
.35	−.46220699424886229565	−.087943553306688133513	−.37735680556076141315
.4	−.46236637072924777344	−.067075631539224926148	−.37271747797639839730
.45	−.45838588104810280947	−.045777188693243047605	−.36563286121393223513
.5	−.45001697146890813343	−.023982773837446344605	−.35584445910606541603
.55	−.43694444222791179535	−.0016150611545678131322	−.34304002442650905123
.6	−.41877692441505661965	.021417671915978278429	−.32684563685890545470
.65	−.39503429791626321840	.045224794037227840313	−.30681645638723853448
.7	−.36513096401300933813	.069937623736922198158	−.28242603273601441995
.75	−.32835278119092827591	.095715398034445688010	−.25305374190333264848
.8	−.28382303116974082780	.12275412910594040701	−.21796900696794990496
.85	−.23044714090374489214	.15130096524457396024	−.17630849139663963600
.9	−.16681210426698377456	.18168005062200866695	−.12703592667444104842
.95	−.090979774570901825562	.21434488601106355137	−.068856253193915809433

Table II

	0	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{4}{9}$	$\frac{5}{9}$	$\frac{2}{3}$	$\frac{7}{9}$	$\frac{8}{9}$	1	$\frac{10}{9}$	$\frac{11}{9}$	$\frac{4}{3}$	$\frac{13}{9}$	$\frac{14}{9}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{17}{9}$	2
ϕ_1	0	$\frac{32\sqrt{2}}{49}$	$\frac{46\sqrt{2}}{49}$	$\frac{6\sqrt{2}}{7}$	$\frac{24\sqrt{2}}{49}$	$\frac{24\sqrt{2}}{49}$	$\frac{6\sqrt{2}}{7}$	$\frac{46\sqrt{2}}{49}$	$\frac{32\sqrt{2}}{49}$	0	0	0	0	0	0	0	0	0	0
ϕ_2	0	$\frac{160\sqrt{2}}{147}$	$\frac{50\sqrt{2}}{147}$	$\frac{10\sqrt{2}}{7}$	$\frac{-20\sqrt{2}}{147}$	$\frac{20\sqrt{2}}{147}$	$\frac{-10\sqrt{2}}{7}$	$\frac{-50\sqrt{2}}{147}$	$\frac{-160\sqrt{2}}{147}$	0	0	0	0	0	0	0	0	0	0
ϕ_3	0	$\frac{17}{147}$	$\frac{-128}{147}$	$\frac{5}{7}$	$\frac{32}{147}$	$\frac{121}{147}$	$\frac{-8}{7}$	$\frac{17}{147}$	$\frac{40}{147}$	3	$\frac{40}{147}$	$\frac{17}{147}$	$\frac{-8}{7}$	$\frac{121}{147}$	$\frac{32}{147}$	$\frac{5}{7}$	$\frac{-128}{147}$	$\frac{17}{147}$	0

Table III

	0	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{3}$	$\frac{4}{9}$	$\frac{5}{9}$	$\frac{2}{3}$	$\frac{7}{9}$	$\frac{8}{9}$
ψ_1	0	$\frac{17\sqrt{2}}{294}$	$\frac{-64\sqrt{2}}{147}$	$\frac{5\sqrt{2}}{14}$	$\frac{16\sqrt{2}}{147}$	$\frac{121\sqrt{2}}{294}$	$\frac{-4\sqrt{2}}{7}$	$\frac{-149\sqrt{2}}{147}$	$\frac{272\sqrt{2}}{147}$
ψ_2	0	$\frac{17\sqrt{6}}{294}$	$\frac{-64\sqrt{6}}{147}$	$\frac{5\sqrt{6}}{14}$	$\frac{16\sqrt{6}}{147}$	$\frac{121\sqrt{6}}{294}$	$\frac{-4\sqrt{6}}{7}$	$\frac{-44\sqrt{6}}{147}$	$\frac{104\sqrt{6}}{147}$
ψ_3	0	$\frac{2\sqrt{129}}{301}$	$\frac{-50\sqrt{129}}{301}$	$\frac{12\sqrt{129}}{43}$	0	0	$\frac{12\sqrt{129}}{43}$	$\frac{-50\sqrt{129}}{301}$	$\frac{2\sqrt{129}}{301}$
ψ_4	0	$\frac{-698\sqrt{43}}{2107}$	$\frac{422\sqrt{43}}{2107}$	$\frac{-60\sqrt{43}}{301}$	$\frac{12\sqrt{43}}{49}$	$\frac{12\sqrt{43}}{49}$	$\frac{-60\sqrt{43}}{301}$	$\frac{422\sqrt{43}}{2107}$	$\frac{-698\sqrt{43}}{2107}$
ψ_5	0	$\frac{586\sqrt{17049}}{39781}$	$\frac{-934\sqrt{17049}}{39781}$	$\frac{60\sqrt{17049}}{5683}$	0	0	$\frac{-60\sqrt{17049}}{5683}$	$\frac{934\sqrt{17049}}{39781}$	$\frac{-586\sqrt{17049}}{39781}$
ψ_6	0	$\frac{-7762\sqrt{5683}}{835401}$	$\frac{8182\sqrt{5683}}{835401}$	$\frac{-808\sqrt{5683}}{39781}$	$\frac{8\sqrt{5683}}{147}$	$\frac{-8\sqrt{5683}}{147}$	$\frac{808\sqrt{5683}}{39781}$	$\frac{-8182\sqrt{5683}}{835401}$	$\frac{7762\sqrt{5683}}{835401}$

	1	$\frac{10}{9}$	$\frac{11}{9}$	$\frac{4}{3}$	$\frac{13}{9}$	$\frac{14}{9}$	$\frac{5}{3}$	$\frac{16}{9}$	$\frac{17}{9}$	2
ψ_1	$-3\sqrt{2}$	$\frac{272\sqrt{2}}{147}$	$\frac{-149\sqrt{2}}{147}$	$\frac{-4\sqrt{2}}{7}$	$\frac{121\sqrt{2}}{294}$	$\frac{16\sqrt{2}}{147}$	$\frac{5\sqrt{2}}{14}$	$\frac{-64\sqrt{2}}{147}$	$\frac{17\sqrt{2}}{294}$	0
ψ_2	0	$\frac{-104\sqrt{6}}{147}$	$\frac{44\sqrt{6}}{147}$	$\frac{4\sqrt{6}}{7}$	$\frac{-121\sqrt{6}}{294}$	$\frac{-16\sqrt{6}}{147}$	$\frac{-5\sqrt{6}}{14}$	$\frac{64\sqrt{6}}{147}$	$\frac{-17\sqrt{6}}{294}$	0
ψ_3	0	0	0	0	0	0	0	0	0	0
ψ_4	0	0	0	0	0	0	0	0	0	0
ψ_5	0	0	0	0	0	0	0	0	0	0
ψ_6	0	0	0	0	0	0	0	0	0	0

Table IV

Dilation Matrices for $n = 3$, s_0, s_1, s_2 Arbitrary

$$C_0 = \begin{pmatrix} \frac{2s_0+1-(p_1+p_2)}{2} & \frac{-(1+3(p_1-p_2)}{6} & 1 \\ \frac{1-(p_1+p_2)}{2} & \frac{6s_0-(1+3(p_1-p_2))}{6} & 1 \\ \frac{(p_1+p_2-1)(s_0-p_1)}{2} & \frac{(p_1-p_2+1/3)(s_0-p_1)}{2} & p_1 \end{pmatrix}$$

$$C_1 = \begin{pmatrix} \frac{2s_1+2-(p_1+p_2+q_1+q_2)}{2} & \frac{-(p_1-p_2+q_1-q_2)}{2} & 1 \\ \frac{p_1+p_2-(q_1+q_2)}{2} & s_1 + 1/3 + \frac{p_1-p_2-(q_1-q_2)}{2} & -1 \\ \frac{(p_1+p_2)(s_1+1-p_2)-s_1-p_1(q_1+q_2)}{2} & \frac{(p_1-p_2)(1/3+s_1-p_2)+s_1/3-p_1(q_1-q_2)}{2} & p_2 \end{pmatrix}$$

$$C_2 = \begin{pmatrix} s_2 + 1/2 - \frac{(q_1+q_2)}{2} & \frac{q_2-q_1+1/3}{2} & 0 \\ \frac{q_1+q_2-1}{2} & \frac{2s_2-1/3+q_1-q_2}{2} & 0 \\ \frac{1-p_1+s_2(p_1+p_2-1)-p_2(q_1+q_2)}{2} & \frac{4p_2-3p_1-1+3s_2(p_1-p_2+1/3)-3p_2(q_1-q_2)}{6} & 1 \end{pmatrix}$$

$$C_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{s_0(q_1+q_2-1)-q_1(p_1+p_2)-q_2+1}{2} & \frac{3s_0(q_1-q_2-1/3)-3q_1(p_1-p_2)+3q_2-4q_1+1}{6} & q_1 \end{pmatrix}$$

$$C_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{(s_1-q_1)(q_1+q_2-1)+q_2(1-p_1-p_2)}{2} & \frac{(s_1-q_1)(q_1-q_2-1/3)-q_2(p_1-p_2+1/3)}{2} & q_2 \end{pmatrix}$$

$$C_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{(s_2-q_2)(q_1+q_2-1)}{2} & \frac{(s_2-q_2)(q_1-q_2-1/3)}{2} & 0 \end{pmatrix}$$